UNIVERSITY EXAMINATIONS, 2018
SECOND YEAR EXAMINATION
FOR
THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

MAT:2111- ORDINARY DIFFERENTIAL EQUATIONS I

Instructions to candidates:

Answer **Question 1** *and* **TWO other Questions**.

All Symbols have their usual meaning.

Remember to use correct English at all times.

DATE: 2018 TIME: 2hrs

Question 1(Entire course) (30 Marks)

(a) Consider the differential equation $\dot{x} = f(x)$ with $f(x)$ given in Figure 1.

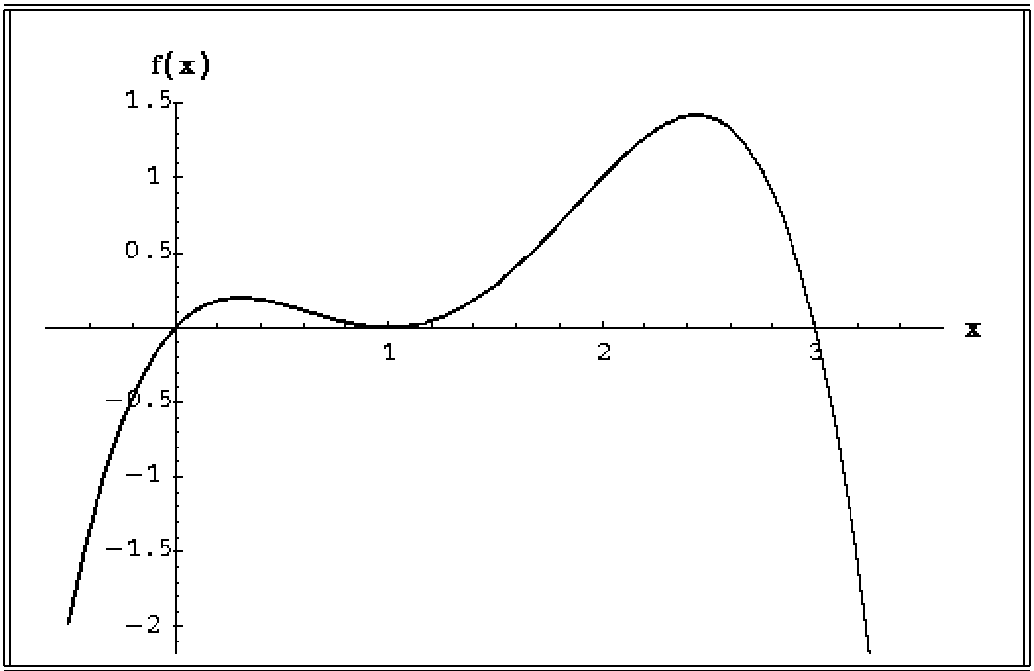

Figure 1: Graph of $(x, f(x))$

(i) Draw the phase line for the differential equation and classify the equilibrium points as sinks, sources,or nodes. **(4 Marks)**
(ii)Give a rough sketch of the slope field for this differential equation, and draw a few solutions into the slope field. **(4 Marks)**
(iii) Consider the solution to the differential equation which satisfies the initial condition:

- $x(0) = 2$. Find

$$\lim_{t\to\infty} x(t)$$

(1 Mark)

- $x(0) = 1$. Find

$$\lim_{t\to\infty} x(t).$$

(1 Mark)

(b) Consider the differential equation

$$\frac{dw}{dt} = (2 - w)\sin w \tag{1}$$

where $w = w(t)$.
Sketch the possible solution curves for (1) with initial conditions $w(0) \in [-\pi\ 2\pi]$

(**Hint: there are nine possible solutions curves**). (**6 Mark**)

(c) Solve the following differential equation:

$$\dot{p} = rp(1 - \frac{p}{K}), \tag{2}$$

where r and K are positive constants. (**4 Marks**)

(d) Solve the differential equation

$$y'' - 5y' + 6y = 0, \tag{3}$$

subject to $y(0) = 1, \; y'(0) = -1$. (**5 Marks**)

(e) From experimental observation it is known that, upto a "satisfactory "approximation, that the surface temperature of an object changes at a rate proportional to its relative temperature. That is, the difference between its temperature and the temperature of the surrounding environment. This is what is known as **Newton's law of cooling**. Thus if, $\theta(t)$ is the temperature of the object at time t, then we have

$$\dot{\theta} = -k(\theta - S), \tag{4}$$

where S is the temperature of the surrounding environment, $k > 0$.

Suppose a corpse was discovered in a motel room at midnight and its temperature was $27°C$. The temperature of the room was kept constant at $16°C$. Two hours later the temperature of the corpse dropped to $24°C$. Find the time of death. (**5 Marks**)

Question 2(Application) (20 Marks)

Consider an elementary model of the learning process of certain list of SMA 208 ODE I formulae. If we let $L(t)$ be the fraction of the list learned at time t, where $L = 0$ corresponds to knowing nothing and $L = 1$ corresponds to knowing the entire list, then we can form a simple model of this type of learning based on the assumption:

• The rate of learning is proportional to the amount left to be learned. Since $L = 1$ corresponds to knowing the entire list, the model is

$$\frac{dL}{dt} = k(1 - L), \tag{5}$$

where k is the constant of proportionality.

(a) For what value of $L, \; 0 \leq L \leq 1$, does learning occur most rapidly? (**1 Mark**)

(b) Suppose two students memorize lists according to the same model:

$$\frac{dL}{dt} = 2(1 - L), \tag{6}$$

(i) If one of the students knows one-half of the list at time $t = 0$ and the other knows none of the list, Explain which student is learning most rapidly at this instant. (**2 Marks**)
(ii) Will the student who starts out knowing none of the list ever catch up to the student who starts out knowing one-half of the list? (**2 Marks**)
(c)Consider two equations showing the rate of memorizing by two students' Juma and Brenda respectively,

$$\frac{dL_J}{dt} = 2(1 - L_J), \tag{7}$$

and

$$\frac{dL_B}{dt} = 3(1 - L_B)^2. \tag{8}$$

(i)Which student has a faster rate of learning at $t = 0$ if they both start memorizing together having never seen the list of formulae before? (**2 Marks**)
(ii)Which student has a faster rate of learning at $t = 0$ if they both start memorizing together having already learned half of the formulae list? (**1 Mark**)
(iii)Which student has a faster rate of learning at $t = 0$ if they both start memorizing together having already learned one-third of the formulae list? (**2 Marks**)
(d)High levels of cholesterol in the blood are known to be a risk factor for heart disease. Cholesterol is manufactured by the body for use in the construction of cell walls and is absorbed from foods containing cholesterol. The following is a very simple model of blood cholesterol levels. Let $C(t)$ be the amount of cholesterol in the blood of a particular person at time t(in milligrams per deciliter). Then

$$\frac{dC}{dt} = k_1(C_0 - C) + k_2E, \tag{9}$$

where
$C_0 =$ the person's "natural" cholesterol level,
$k_1 =$ "production" parameter,
$E =$ amount of cholesterol eaten (per day), and
$k_2 =$ "absorption" parameter.

(i) Suppose $C_0 = 200, k_1 = 0.1, k_2 = 0.1, E = 400$, and $C(0) = 150$. What will the person's cholesterol level be after 5 days on this diet? (**2 Marks**)

(ii) What will the person's cholesterol level be after a long time on this diet? (**1 Marks**)

(iii) Suppose that, after a long time on the high cholesterol diet described above, the person goes on very low cholesterol diet, so E changes to $E = 100$. (The initial cholesterol level at the starting time of this diet is the result of part (d)(ii).) What will the person's cholesterol level be after a 1 day on the new diet, after a very long time on the new diet? (**4 Marks**)

(iv) Suppose the person stays on the high cholesterol diet but takes drugs that block some of the uptake of cholesterol from food, so k_2 changes to $k_2 = 0.075$. With the cholesterol level from part (iii), what will the person's cholesterol level be after a very long time? (**3 Marks**)

Question 3 (Mixed Grill :20 Marks)

(a) Find the integral curves of the equation

$$\frac{dx}{y(x+y)+az} = \frac{dy}{x(x+y)-az} = \frac{dz}{z(x+y)}. \tag{10}$$

(**5 Marks**)

(b) Solve the initial value problem

$$y'' + 2y' + y = e^{-x}, \quad y(0) = y'(0) = 1.$$

(**5 Marks**)

(c) Find a particular solution to the equation

$$y''' - 4y' = x + 3\cos x.$$

(**5 Marks**)

(d) Find the fifth degree Taylor polynomial of the solution to the differential equation

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{11}$$

(**5 Marks**)

Question 4 (Bifurcation : 20 Marks)

(a) Consider the constant harvest rate model of fish given below.

$$\dot{p} = rp(1 - \frac{p}{N}) - E, \tag{12}$$

where r is the growth rate parameter, $p := p(t)$ is the population of fish at time t, N is the carrying capacity of the area, and $E > 0$ is a constant harvest rate. How does the population of fish vary as E is increased? We answer this question via the following little questions.

(i) What will be the long-term population when harvesting is banned; that is, $E = 0$ (Justify your answer)? (**2 Marks**)

(ii) Find the fixed points of Equation (12) for positive values of E and show whether they are stable or unstable. (**4 Marks**)

(iii) Sketch, on the same grid, the graph of $f(p, E) := rp(1 - \frac{p}{N}) - E$ against P for $E \in [0, \frac{rN}{4} + \epsilon)$, $\epsilon > 0$. What do you notice about the separation and the number of the equilibrium points? (**4 Marks**)

(iv) Using $E \geq 0$ as a parameter, determine the bifurcation value and sketch the bifurcation diagram. On the bifurcation diagram indicate all possible phase lines. (**4 Marks**)

(v) What is likely to happen to the fish population when $E > \frac{rN}{4}$. (**1 Mark**)

(b) Consider the population model

$$\dot{p} = 2p - \frac{p^2}{50}, \tag{13}$$

for a species of fish in a lake. Time is measured in years. Suppose it is decided that fishing will be allowed in the lake, but it is unclear how many fishing licenses should be issued. Suppose the average catch of a fisherman with a license is 3 fish per year(these are hard fish to catch). [*Hint;* To answer the following questions use Equations (12) and (13) and the results in (a).]

(i) What is the largest number of licenses that can be issued if the fish are to have a chance to survive? (**2 Marks**)

(ii) Suppose the number of fishing licenses in part (b)(i) is issued. What will happen to the fish population; that is, how does the behaviour of the population depend on the initial population? (**3 Marks**)