MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

SCHOOL OF PURE, APPLIED AND HEALTH SCIENCES

MASTER OF SCIENCE EXAMINATION

COURSE CODE: MAT 8107

COURSE TITLE: OPERATOR THEORY I

DATE: **TIME: 3 Hours**

INSTRUCTIONS TO CANDIDATES

Answer Question **ONE** and any other **TWO** questions

*This paper consists of **THREE** printed pages. Please turn over.*

QUESTION ONE – 30 MARKS

a) State the Lax-Milgram Lemma. **(2 Marks)**

b) i) What is a linear projection operator? **(1 Mark)**

ii) Given that H is a Hilbert space, prove that $P = P_1P_2$ is a projection on H if and only if $P_1P_2 = P_2P_1$. **(4 Marks)**

c) Prove that the spectrum of a bounded self adjoint linear operator $T : H \to H$ on a complex Hilbert space H is real. **(5 Marks)**

d) Define the spectrum of an operator T hence find the spectrum of the matrix $M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for $b \neq 0$. **(5 Marks)**

e) Given that $T \in B(X, X)$ where X is a Banach space, prove that if $\|T\| < 1$, then $(I - T)^{-1}$ exists as a bounded linear operator on X and $(I - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + \cdots$

(5 Marks)

f) Let $T : H \to H$ be a bounded positive self adjoint operator on a complex Hilbert space H

i) Give a precise definition of square root of a positive operator T. **(1 Mark)**

ii) Using the positive square root of T, show that for all $x, y \in H$,

$$\left|\langle Tx, y\rangle\right| \leq \langle Tx, x\rangle^{\frac{1}{2}} \langle Ty, y\rangle^{\frac{1}{2}}$$ **(4 Marks)**

g) Find a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is idempotent but not self adjoint. **(3 Marks)**

QUESTION TW0 – 15 MARKS

a) If S is a bounded linear operator on a Banach space X and $\|S\| < |\lambda|$, $S_\lambda = (\lambda I - S)^{-1}$ is a bounded operator. Prove that $S_\lambda = \sum_{n=0}^{\infty} \frac{S^n}{\lambda^{n+1}}$ **(5 Marks)**

b) i) State Fredholm Equation. **(1 Mark)**

ii) Given the integral operator $T : L_2[0, 2\pi] \to L_2[0, 2\pi]$ defined by

$(Tu)(\psi)=\int_0^{2\pi}\cos(\psi-y)u(y)\,dy$. Prove that T has exactly one non zero eigen value $\lambda=\pi$ and the corresponding eigen function $u(\psi)=\alpha\cos\psi+\beta\sin\psi$ where α and β are arbitrary constants. **(6 Marks)**

c) Given that P_1 and P_2 are projections on a Hilbert space H. Prove that if $\|P_1x\|\le\|P_2x\|$, then $P_1\le P_2$. **(3 Marks)**

QUESTION THREE – 15 MARKS

a) If a sum $P_1+P_2+\cdots+P_k$ of projections $P_j:H\to H$ (H a Hilbert space) is a projection, show that $\|P_1x\|^2+\|P_2x\|^2+\cdots\|P_kx\|^2\le\|x\|^2$. **(5 Marks)**

b) Given that T is a normal operator, prove that $Tx=\lambda x$ if and only if $T^*x=\bar{\lambda}x$.

(5 Marks)

c) Let $T:H\to H$ be a bounded self adjoint linear operator on a complex Hilbert space H. Prove that the residual spectrum of T is empty. **(5 Marks)**

QUESTION FOUR – 15 MARKS

a) Prove that the spectral radius and the norm of a self adjoint operator T on X coincide.

(4 Marks)

b) Given that $Q:H\to H$ $\left(Q=S^{-1}PS\right)$ where S and P are bounded and linear. If P is a projection and S is unitary, then show that Q is a projection. **(4 Marks)**

c) Let $T:H\to H$ be a bounded self adjoint linear operator on a complex Hilbert space H, then show that the eigen space of T associated with distinct eigen values are orthogonal.

(3 Marks)

d) Suppose $T:X\to X$ is a compact linear operator on a normed space X and $\lambda\ne0$. Then show that $Tx-\lambda x=y$ has a solution x if $f(y)=0$ for all $f\in X^*$ satisfying $T^*f-\lambda f=0$. **(4 Marks)**

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