UNIVERSITY EXAMINATIONS, 2023
FIRST YEAR EXAMINATION
FOR
THE DEGREE OF MASTER OF SCIENCE
8219:- :PARTIAL DIFFERENTIAL EQUATIONS II
Instructions to candidates:

Answer Question ONE and two Other Questions. All Symbols have their usual meaning

DATE: April 2023 TIME:9.00 a.m. to 12.00 NOON

Question 1:Entire syllabus (30 Marks)

(a) **(Linear Evolution equations)** Assume $U \subset \mathbb{R}^n$ is an open, bounded set, with smooth boundary, and $T > 0$. Prove there is at most one smooth solution of the IBVP (1) for the heat equation with Neumann boundary conditions

$$\begin{aligned} u_t - \Delta u &= f \text{ in } U_T, \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \times [0, T], \\ u &= g \text{ on } U \times \{t = 0\}. \end{aligned} \tag{1}$$

(5 Marks)

(b) Assume U is connected. Use (i) energy methods and (ii) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{aligned} -\Delta u &= 0 \text{ in } U, \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \end{aligned} \tag{2}$$

are $u = C$, for some constant C. **(8 Marks)**

(c) Assume $u \in H^1(U)$ is a bounded weak solution of

$$-\text{div}(A(x)Du) = 0 \text{ in } U, \tag{3}$$

where $A(x) \in L^\infty(U)$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and smooth, and set $w = \phi(u)$. Show that w is a weak subsolution; that is,

$$B[w, v] \le 0 \;\; \forall \;\; v \in H_0^1(U), \quad v \ge 0.$$

(3 Marks)

(d) Consider the Hamilton-Jacobi equation:

$$\begin{aligned} u_t + H(Du) &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t = 0\}, \end{aligned} \tag{4}$$

where $u : \mathbb{R}^n \times [0, \infty)$ is the uknown, $u = u(x, t)$, and $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$, and $H : \mathbb{R}^n \to \mathbb{R}$, the *Hamiltonian*, and $g : \mathbb{R}^n \to \mathbb{R}$ are given. Let $L = L(v, x)$, $(v, x) \in \mathbb{R}^n \times \mathbb{R}^n$ be the *Lagrangian*. Set $\dot{w}(s) := v$, $w(s) := x$. For fixed $x, y \in \mathbb{R}^n$, define the *action* functional:

$$I[w(.)] := \int_0^t L(\dot{w}(s), w(s))ds, \tag{5}$$

where the dot denotes the derivative with respect to s. Let w in (5) belong to the *admissible* set

$$\mathcal{A} := \{w(.) \in C^2([0,t];\mathbb{R}^n) | w(0) = y, w(t) = x\}.$$

The basic problem is to find a curve $x(.) \in \mathcal{A}$ satisfying

$$I[x(.)] = \min_{w(.) \in \mathcal{A}} I[w(.)]. \tag{6}$$

Proof the function $x(.)$ solves the system of Euler-Lagrange equations

$$-\frac{d}{ds}(D_v L(\dot{x}(s), x(s))) + D_x L(\dot{x}(s), x(s)) = 0, \quad 0 \leq s \leq t.$$

(**8 Marks**)

(e) **(Lax-Milgram Theorem)** *Assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ *such that*

(i) $|B[u,v]| \leq \alpha ||u|| ||v||, \quad (u, v \in H)$ *and*

(ii) $\beta ||u||^2 \leq B[u,u], \quad u \in H.$

Finally, let $f : H \to \mathbb{R}$ *be a bounded linear functional on* H*. Then there exists a unique element* $u \in H$ *such that*

$$B[u,v] = < f, v >$$

for all $v \in H$.

Use the Lax-Milgram theorem to prove the following: There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha ||u||_{H_0^1(U)} ||v||_{H_0^1(U)},$$

and (**3 Marks**)

$$\beta ||u||^2_{H_0^1(U)} \leq B[u,u] + \gamma ||u||^2_{L^2(U)}$$

for all $u, v \in H_0^1(U)$ (**3 Marks**)

Question 2 (20 Marks) (Second-order Elliptic Equations) .

(a) Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in W^{1,p}(U)$ for some $1 \leq p < n$. Prove that $u \in L^q(U)$, with estimate

$$||u||_{L^q(U)} \leq C||Du||_{L^p(U)}, \qquad (\textbf{3 Marks})$$

for each $q \in [1, p^*]$, the constant C depends only on p, q, n, and U. In particular, $\forall \ \ 1 \leq p \leq \infty$,

$$||u||_{L^p(U)} \leq C||Du||_{L^p(U)}. \qquad (\textbf{2 Marks})$$

(b) Hence show that for $u \in H_0^2(U)$,

(i)

$$||u||_{L^2(U)} \leq C_1||Du||_{L^2(U)}, \qquad (\textbf{1 Mark})$$

(ii)

$$||Du||_{L^2(U)} \leq C_2||D^2u||_{L^2(U)}, \qquad (\textbf{2 Marks})$$

where C_1, C_2 are constants. The Poincar'e Inequality for $H_0^2(U)$- norm

(iii)

$$||u||_{H_0^2(U)} \leq C_3||D^2u||^2_{L^2(U)}, \qquad (\textbf{2 Marks})$$

(iv)

$$||\Delta u||_{L^2(U)} = |D^2u||^2_{L^2(U)}, \qquad (\textbf{2 Marks})$$

where C_3 is a constant and

(v)

$$||u||_{H_0^2(U)} \leq |\Delta u||^2_{L^2(U)}, \forall \ u \in H_0^2(U) \qquad (\textbf{2 Marks})$$

(c) For a given $f \in L^2(U)$, prove that the function $u \in H_0^2(U)$ is a unique weak solution of the boundary-value problem for the *biharmonic equation*

$$\begin{aligned} \Delta^2 u \ &= f, \qquad \text{in } \ U; \\ u = 0 = \frac{\partial u}{\partial \nu} \ &= 0, \qquad \text{on } \ \partial U, \end{aligned}$$

provided

$$\int_U \Delta u \Delta v dx = \int_U f v dx$$

for all $v \in H_0^2(U)$. (**6 Marks**)

Question 3 (20 Marks) (Application of Semigroup to PDEs)
Consider first the parabolic initial/boundary-value problem

$$\begin{aligned} u_t + Lu &= 0 \text{ in } U_T \\ u &= 0 \quad \text{on } \partial U \times [0,T] \\ u &= g \text{ on } U \times \{t=0\}, \end{aligned} \tag{7}$$

We assume $Lu = -div(A(x)Du) + b(x).Du + c(x)u$ and satisfies the usual strong ellipticity condition, and has smooth coefficients, which do not depend on t. We additionally suppose that the bounded open set U has a smooth boundary. We propose to reinterpret Equation (7) as the flow determined by a semigroup on $X = L^2(U)$. Set $\mathcal{D}(A) := H_0^1(U) \cap H^2(U)$ and define $Au := -Lu$, if $u \in \mathcal{D}(A)$. You will need the *Hille-Yosida Theorem* and the energy estimates

$$\beta||u||^2_{H_0^1(U)} \leq B[u,u] + \gamma||u||^2_{L^2(U)}$$

for constants $\beta > 0, \gamma \geq 0$, where $B[,]$ is the bilinear form associated with L.

(a) Show that $\mathcal{D}(A)$ is dense in X **(1 Mark)**

(b) Prove that the Operator A is closed **(4 Marks)**

(c) Recall

$$\begin{aligned} \lambda u + Lu &= f \text{ in } U \\ u &= 0 \quad \text{on } \partial U \text{ on } U, \end{aligned} \tag{8}$$

has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$ provided $\lambda \geq \gamma$. Prove that the resolvent operator $R_\lambda := (\lambda I - A)^{-1}$ satisfies

$$||R_\lambda|| \leq \frac{1}{\lambda - \gamma}, \quad \lambda > \gamma$$

(10 Marks)

(d) Hence show that Equation (7) has a γ-contraction semigroup. **(5 Marks)**

Question 4 (20 Marks) (Calculus of Variations)
Let $u \in \mathbb{R}^n$ be a bounded open set with Lipschitz boundary.

(a) Show there exists a unique minimizer $u \in \mathcal{A}$ of

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - fwdx,$$

where $f \in L^2(U)$ and

$$\mathcal{A} := \{w \in H_0^1(U) | |Dw| \leq 1 \quad a.e.\}.$$

(**10 Marks**)

(b) Prove

$$\int_U Du.D(w-u)dx \geq \int_U f(w-u)dx,$$

for all $w \in \mathcal{A}$. (**10 Marks**)

The final Mark of this paper will be out of 60%