UNIVERSITY EXAMINATIONS, 2023
FIRST YEAR EXAMINATION
FOR
THE DEGREE OF MASTER OF SCIENCE

MAT 8211- Applied Dynamical systems I

Instructions to candidates:

Answer Question ONE and any other Two Questions. All Symbols have their usual meaning

DATE: 2023 **TIME:**. 9.00 A.M. To 12.00 A.M.

Question 1 (30 Marks)

(a) Consider an autonomous nonlinear system of differential equations,

$$\dot{x} = f(x, \mu), \tag{1}$$

with $x \in U \subset \mathbb{R}^n$, $\mu \in V \subset R$ where U and V are open sets in $\mathbb{R}^n$ and $\mathbb{R}$, respectively. μ is a parameter. Explain clearly how you may completely analyse it. *You have, among other aspects include, normal form, codimension, rescaling, local and global bifurcations.* (**6 Marks**)

(b) **(Saddle-node bifurcation in ecology)** Consider the following differential equation, which models a single population under constant harvest:

$$\dot{x} = rx(1 - \frac{x}{K}) - \mu, \tag{2}$$

where x is the population number; r and K are the *intrinsic growth rate* and the *carrying capacity* of the population, respectively, and μ is the *harvest rate*, which is a control parameter. Find a parameter value μ_0, at which the system has a Saddle-node bifurcation, and check the generacity conditions. Sketch a bifurcation diagram in the (μ, x)-plane and indicate on it the direction of the flow with the types of stability. Based on the analysis, explain what might be a result of overharvesting on the ecosystem dynamics. Is the bifurcation catastrophic in this example? (**7 Marks**)

(c) **(Neimark-Sacker bifurcation in the delayed logistic equation)** Consider the following recurrence equation:

$$u_{k+1} = ru_k(1 - u_{k-1}). \tag{3}$$

This is a simple population dynamics model, where u_k stands for the density of a population at time k, and r is the growth rate. It is assumed that the growth is determined not only by the current population density but also by its density in the past, u_{k-1}. Let $v_k := u_{k-1}$. Rewrite the difference equation in the form

$$\begin{aligned} u_{k+1} &= f_1(u_k, v_k; r), \\ v_{k+1} &= f_2(v_k). \end{aligned} \tag{4}$$

Hence write down the two-dimensional system discrete-time dynamical system in the form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, x_2; r) \\ f_2(x_1) \end{pmatrix}, \tag{5}$$

where $x_1 := u_k, x_2 := v_k$. Find the nontrivial fixed point (x_1, x_2), $x_1 = x_2$ and the value of r at which we have a Neimark-Sacker bifurcation. Give a detailed analysis of your working. **(7 Marks)**

(d) **(Just Elementary facts)** Consider the planar differential equation below;

$$\begin{aligned} \dot{x} &= x(150 - x - 3y), \\ \dot{y} &= y(100 - 2x - y), \quad (x, y) \in R^2. \end{aligned} \tag{6}$$

- Draw the $x-$ and $y-$ nullclines for this system. **(2 Marks)**
- Find all equilibria points for this system and sketch their locations on your picture of nullclines. **(4 Marks)**
- Indicate by little arrows the direction of the vector field in all regions bounded by the nullclines. **(2 Marks)**
- What can you say about the fate of the solution curve beginning at $(60, 20)$? Sketch this solution curve. **(1 Marks)**
- What about the fate of the solution beginning at $(20, 20)$? What are the possible fates for this solution? **(1 Marks)**

Question 2 (15 Marks)(Insect (Spruce Budworm)-outbreak model)

Below is a model for the spruce budworm population at time t, $x := x(t)$ surviving on a forest foliage of size $y := y(t)$:

$$\begin{aligned} \dot{x} &= rx(1 - \tfrac{x}{y}) - \tfrac{Fx^2}{L^2+x^2} =: f(x, y), \\ \dot{y} &= \lambda - \alpha x =: g(x, \lambda), \end{aligned} \tag{7}$$

where F, L, λ, α are positive constants. By studying the system above do the following:

(a) Verify that $f_y(x, y) > 0$ and $g_x(x, \lambda) < 0$ for all $x, y > 0$. **(2 Marks)**

(b) Show that there exists a nonnegative function $\varphi(x)$ for $x > 0$ such that $f(x, \varphi(x)) = 0$, and find this function. **(2 Marks)**

(c) Show that there is a function $\bar{x} := \bar{x}(\lambda) > 0$ with $g(\bar{x}(\lambda), \lambda) = 0$ and $g_x(\bar{x}(\lambda)) < 0$. **(2 Marks)**

(d) Show that the stationary point $(\bar{x}, \varphi(\bar{x}))$ is stable when $\varphi'(\bar{x}) > 0$ and unstable when $\varphi'(\bar{x}) < 0$; the prime denotes differentiation with respect to x. (**3 Marks**)

(e) Let $\xi := (\bar{x}(\hat{\lambda}), \varphi(\bar{x}(\hat{\lambda})))$ be a Poincaré-Andronov- Hopf (PAH) bifurcation point for Equation(7) above. Prove that a necessary condition for a PAH bifurcation to occur at ξ is

$$f_{xx}(\xi) \neq 0, g_{\lambda}(\xi) \neq 0. \qquad \textbf{(3 Marks)}$$

(f) Sketch the bifurcation diagram in the x, y-plane. Indicate on the diagram where the flow is stable, unstable, and the PAH bifurcation points. (**3 Marks**)

Question 3 (15 Marks) (Center Manifold) Consider the system of differential equations below:

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1 y, \\ \dot{x}_2 &= x_1 + x_2 y, \\ \dot{y} &= -y - x_1^2 - x_2^2 + y^2. \end{aligned} \tag{8}$$

(a) By studying the system in Equation(8) above, show that there is a local center manifold of $(x_1, x_2, y) = 0$ that can be written as a graph $y = h(x_1, x_2)$ for $|x_1| < \delta$, and $|x_2| < \delta$ for a small $\delta > 0$. (**4 Marks**)

(b) By using an appropriate Taylor series approximation to $h(x_1, x_2)$, about the origin, determine the equations governing the flow on the center manifold. (**4 Marks**)

(c) By expressing the governing equations determined in (b) above in polar coordinates, sketch the phase portrait for the system (8) near the origin. (**5 Marks**)

(d) Is the origin stable or unstable? State your reason(s) for the choice of your answer. (**2 Marks**)

Question 4 (15 Marks) (Synchronization of fully coupled systems) Consider the following system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \end{aligned} \tag{9}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Suppose the system is dissipative, then there exists a global attractor $\mathcal{A}$. The system (9) is *synchronized* for y with respect to x, if there exists $\mathcal{C}^1$ map $H : \mathbb{R}^n \to \mathbb{R}^m$ such that the graph of H, denoted by $graph(H)$, is invariant and globally attracting.

(a) What does the statement "$graph(H)$, is invariant and globally attracting" mean? (**2 Marks**)
What is the map H in case $m = n$ in (9); that is, *mutual synchronization*? (**1 Mark**)

(b) Suppose, for (9), there exist an invariant manifold $M = graph(H)$.
Using the *generalized Lyapunov exponents* clearly explain how you would investigate the robustness of this manifold. (**6 Marks**)

(c) Let

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{10}$$

be a dissipative process and $\mathcal{A}$ be the attractor. Denote λ_M, λ_m the maximal and minimal Lyapunov exponents over $\mathcal{A}$. Consider the coupled system

$$\begin{aligned} \dot{x} &= kI(y-x) + f(x) \\ \dot{y} &= kI(x-y) + f(y), \end{aligned} \tag{11}$$

where I in an identity matrix of order n and $k > 0$ is a constant. The diagonal $M = \{(x, y) : x = y\}$ is invariant under (11), and the coupled system is synchronized if M is attracting.

(i) With the change of variables defined thus,

$$u = \frac{y-x}{2}, \quad v = \frac{y+x}{2}.$$

Find an expression of the system (11) in these coordinates. (**2 Marks**)

(i) Synchronization is equivalent to $\{u = 0\}$ is attracting. Linearize the system obtained in (i) about M; that is, about $u = 0$, $v = v_0(t)$ and investigate the robustness of the manifold M using the generalized lyapunov exponents and λ_M, λ_m. (**4 Marks**)

Question 5 (15 Marks) (Local Bifurcations of a Vector Field) Consider the following class of feedback control systems

$$\begin{aligned} \ddot{x} + \delta\dot{x} + g(x) &= -z, \\ \dot{z} + \alpha z &= \alpha\gamma(x - r), \end{aligned} \tag{12}$$

where x and $\dot{x}$ represent the displacement and the velocity, respectively, of an oscillatory system with nonlinear stiffness $g(x) := x(x^2 - 1)$ and linear damping $\delta\dot{x}$ subject to negative control z. The controller has first-order dynamics with time constant $\frac{1}{\alpha}$ and gain γ. A constant or time-varying bias r can be applied. This system provides the simplest possible model for a nonlinear elastic system whose position is controlled by a servomechanism with negligible inertia. Rewriting (12) as a system of first order equations we obtain:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \delta y - z, \qquad (x, y, z) \in \mathbb{R}^3 \\ \dot{z} &= \alpha\gamma x - \alpha z, \end{aligned} \tag{13}$$

where $\delta, \alpha, \gamma > 0$, and $r = 0$.

(a) Show that (13) has fixed points at

$$(x, y, z) = (0, 0, 0) =: E_0,$$

and

$$(x, y, z) = (\pm\sqrt{1-\gamma}, 0, \pm\gamma\sqrt{1-\gamma}) =: E_{\pm}, \quad (\gamma < 1).$$ (**2 Marks**)

(b) Linearize about these three fixed points and **derive the expressions for** γ**, shown below,** so that (13) has the following **bifurcation surfaces** in (α, δ, γ) space.
(i) For

$$\gamma = 1,$$

one eigenvalue is zero for E_0. (**2 Marks**)
(ii) For

$$\gamma = \frac{\delta}{\alpha}(\alpha^2 + \alpha\delta - 1), \qquad \gamma > 1,$$

a pair of eigenvalues is pure imaginary for E_0. (**2 Marks**)
(iii) For

$$\gamma = \frac{\delta}{\alpha + 3\delta}(\alpha^2 + \alpha\delta + 2), \qquad 0 < \gamma < 1,$$

a pair of eigenvalues is pure imaginary for $E_{\pm}$. (**3 Marks**)
(c) Show that these three surfaces meet on the curve

$$\gamma = 1, \quad \delta = \frac{1}{\alpha},$$

where there is a double-zero eigenvalue with the third eigenvalue being

$$\frac{-(1+\alpha^2)}{\alpha}.$$

(**2Marks**)

(d) Fix $\alpha > 0$. Briefly describe how one can go about studying the bifurcations from the double-zero eigenvalue in (δ, γ) plane. (*You are expected to mention, the normal form, center manifold, the codimension of the fixed point, universal unfoldings among other tools for studying nonhyperbolic fixed points).* (**4 Marks**)

Question 5 (15 Marks) **(Invariant Manifolds)** Consider an autonomous system of ordinary differential equations in n-dimensional Euclidean space

$$\dot{x} = f(x), \quad f \in \mathcal{C}^r(U, \mathbb{R}^n), \tag{14}$$

where U is an open set in $\mathbb{R}^n$. Let $\bar{M} \subset \mathbb{R}^n$ be a compact overflowing invariant manifold of this system and let $\varphi_t(x)$ denote the flow generated by Equation (14).

(a) Explain what we mean when one says that M is a normally hyperbolic invariant manifold for $\varphi_t(x)$. (**4 Marks**)

(b) State clearly what the generalized *Lyapunov-type numbers* for Equation(14) are.(**4 Marks**)

(c) By use of the Lyapunov-type numbers, state the conditions necessary for the manifold M to be stable and persist under small perturbations of the vector field in Equation(14) (**2 Marks**)

(d) Consider the vector field given by:

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \\ \dot{y} &= By + g(x,y) \qquad (x,y) \in \mathbb{R}^u \times \mathbb{R}^s, \end{aligned} \tag{15}$$

where $f(0,0) = g(0,0) = Df(0,0) = Dg(0,0) = (0,0)$. A and B are constant matrices and we denote the eigenvalues of A and B by $\lambda_1, \lambda_2, \ldots, \lambda_u$ and $\mu_1, \mu_2, \ldots, \mu_s$ respectively. Furthermore assume that

$$Re\lambda_1 \geq Re\lambda_2 \geq \cdots \geq Re\lambda_u > 0 > Re\mu_1 \geq Re\mu_2 \geq \cdots \geq Re\mu_s.$$

(i) Find the overflowing invariant manifold M for the linearized system associated with the vector field (15) (**1 Mark**)

(ii) Determine the generalized Lyapunov-type numbers for the flow defined on the manifold $\bar{M}$. (**2 Marks**)

(iii) Based on the type numbers in (ii) and for x, y small in (15), what is the relationship between the manifold of the linearized system, M, and that of (15)? (**2 Marks**)