UNIVERSITY EXAMINATIONS, 2023
FIRST YEAR EXAMINATION
FOR
THE DEGREE OF MASTER OF SCIENCE

MAT 8210:-Ordinary differential Equations II

Instructions to candidates:

Answer any Three Questions. All Symbols have their usual meaning

DATE: TIME:

Question 1 (20 Marks)

Given a second order scalar equation

$$\ddot{u} + u = cos\omega t, \quad \omega > 0 \tag{1}$$

(i) Let $x_1 = u$ and find its equivalent first order system. (**3 Marks**)

(ii) Find the adjoint equation for the system in (i) and determine its general solution. (**5 Marks**)

(iii) Show that for $\omega \neq 1$, the adjoint equation has no trivial solution of least period $\frac{2\pi}{\omega}$ which satisfies the Fredholm's alternative

$$\int_0^T y(t)f(t)dt = 0$$

, hence (1) has a unique periodic solution of period $\frac{2\pi}{\omega}$. Whereas if $\omega = 1$, the adjoint equation has period 2π, which doesn't satisfy the Fredholm's alternative thus (1) has no periodic solution of period 2π (**12 Marks**)

Question 2 (20 Marks)

The equation of a pendulum with vertically oscillating sinusoidal support can be written as

$$\ddot{\theta} + c\,\dot{\theta} + \frac{g - R\omega^2 sin\omega t}{l} sin\theta = 0$$

If $\tau = \omega t\,,\ \frac{R}{l} = \varepsilon\,,\ \ \frac{\varepsilon^2}{\kappa^2} = \frac{g}{l\omega^2}\,,\ \frac{c}{\omega} = 2\varepsilon\alpha$
the equation becomes

$$\theta'' + 2\varepsilon\alpha\theta' + (\varepsilon^2\ \kappa^2 - \varepsilon sin\tau)sin\theta = 0$$

with " $'$ " $= \frac{d}{d\tau}$

(i) Write the equivalent first order system for the 2^{nd} order equation in τ (**3 Marks**)

(ii) For ε small, show that the system has periodic solutions of period $\frac{2\pi}{\omega}$ if θ is in the neighbourhood of 0, π or 2π and $cos^{-1}(-2\kappa^2)$ (**7 Marks**)

(iii) By introducing variables ϕ and Ω as

$$\begin{aligned}\theta &= \phi - \varepsilon\ sin\tau\ sin\phi\\ \theta' &= \varepsilon\ \Omega - \varepsilon\ cos\tau\ sin\phi\end{aligned}$$

Use averaging to determine the stability property as functions of α and κ. (**10 Marks**)

Question 3 (20 Marks)
Given the Van der pol equation

$$\ddot{u} - \varepsilon\ (1 - u^2)\dot{u} + u = 0$$

with $\varepsilon > 0$ a small parameter

(i) Let $x_1 = u$ and find its equivalent first order system. (**4 Marks**)

(ii) For $\varepsilon = 0$, find the general solution of the system in (i) and show that it's a unique periodic orbit. (**4 Marks**)

(iii) Suppose g(x,y) has continous first derivatives with respect to x, y. Show that there is an $\varepsilon_o > 0$ such that for $|\varepsilon| < \varepsilon_o$ the equation

$$\ddot{x} - \kappa\ (1 - x^2)\dot{x} + x = \varepsilon\ g(\dot{x}, x), \kappa > 0$$

has a unique periodic solution in the neighbourhood of the periodic orbit in (ii). Show that this orbit is asymptotically orbitaly stable with asymptotic phase. (**12 Marks**)

Question 4 (20 Marks)
Given the equation

$$\begin{aligned}\dot{x_1} &= x_2\\ \dot{x_2} &= -x_1 + \varepsilon(1 - x_1^2)x_2 + Asin\omega t\end{aligned}$$

(a) Discuss the existence of integral manifolds of the equation **(10 Marks)**

(b) For ε small and various values of A and ω, let

$$\begin{aligned} x_1 &= \rho sin\theta_1 + A(1-\omega^2)^{-1} sin\omega t \\ x_2 &= \rho cos\theta_1 + A\omega(1-\omega^2)^{-1} cos\omega t \\ \theta_2 &= t \end{aligned}$$

(i) Obtain a system of differential equations for θ_1, θ_2 and ρ **(4 Marks)**

(ii) Discuss the stability property of the manifold given the averaged equation in b(i) above **(6 Marks)**

Question 5 (20 Marks)
Given the nonlinear system

$$\begin{aligned} \dot{x} &= -y + xz^2 \\ \dot{y} &= x + yz^2 \\ \dot{z} &= -z(x^2 + y^2) \end{aligned}$$

(a) Show that this system has a periodic orbit given by $\gamma(t) = \begin{pmatrix} cost \\ sint \\ 0 \end{pmatrix}$ or $\gamma(t) = (cost, sint, 0)^T$ **(4 Marks)**

(b) Find the linearisation of this system about Y(t) **(4 Marks)**

(c) Find the fundamental matrix $\Phi(t)$ of the autonomous linear system in (b) which satisfies $\Phi(0) = I$ **(4 Marks)**

(d) Find the characteristics exponents and multipliers of $\gamma(t)$ **(4 Marks)**

(e) Determine the dimensions of the stable, unstable and center manifolds of $\gamma(t)$ **(4 Marks)**