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On Skew Quasi-P-Class (Q) Operators

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Abstract

Skew quasi-p-class (Q) operator is introduced. We show that this class satisfies Bishop's property. We equally show that this class is isoloid and polaroid. Results linking this class to other classes such as class (Q) are also given and a result showing that this class doesn't preserve similarity of operators is outlined.

Keywords: Quasi-p-normal; CLass (Q); normal operators; Skew quasi-p-class (Q).

1. Introduction

H represents the usual Hilbert space in this paper. All operators are bounded unless otherwise. $\sigma(S)$ is the spectrum of S while $\sigma_p(S)$ its point spectrum. Senthilkumar [1] studied $\mathcal{G}^*\mathcal{G}(\mathcal{G}+\mathcal{G}^*) = (\mathcal{G}+\mathcal{G}^*)\mathcal{G}\mathcal{G}^*$ and $\mathcal{G}^*\mathcal{G}^n(\mathcal{G}+\mathcal{G}^*) = (\mathcal{G}+\mathcal{G}^*)\mathcal{G}^n\mathcal{G}^*$ operators. They characterized these classes in terms of composition, composite multiplication and weighted composition operators. Jibril [4] introduced the class of (Q) operators in 2010. Jibril [4] studied interesting basic properties that this class gets to enjoy. In particular, this class was linked to the classes of quasinormal operators, θ-operators, Isometry operators and the class of normal operators. Paramesh [9] later expanded the class (Q) operator to n-power class (Q) operators. Manikandan [6] later echoed this class by expanding to $\mathcal{G}^{*2}\mathcal{G}^{n+k} = (\mathcal{G}^*\mathcal{G}^{n+k})^2$ for n which is positive definite and $0 \leq k$. New theorems were characterized for this class, in particular, it was shown that if a bounded operator $\mathcal{G}$ satisfies $\mathcal{G}^*\mathcal{G}^{n+k} = \mathcal{G}^{n+k}\mathcal{G}^*$, $\mathcal{G}$ is $(n+k)$-power class (Q). Later on Revathi [11] extended the quasi class (Q) into M-quasi class (Q). Their properties were investigated.

In particular, results showed that the sum of two M quasi class (Q) and the product of two M quasi class (Q) is still M quasi class (Q). Wanjala Victor and Nyongesa [13] took the study of class (Q) operators to (α,β)-class (Q) for $0 \leq \alpha \leq 1 \leq \beta$. They investigated nice algebraic operations of this class, for instance, it was shown that if $\mathcal{G}$ is (α,β)-class (Q), then $\mathcal{G}^*$ is similarly (α,β)-class (Q). Class of (α,β)-class (Q) [13] was also linked to other classes such as the (α,β)-normal operator. K^* Quasi-n-class (Q)

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operators were covered in [14] where basic properties of this class were highlighted. Wanjala Victor and Beatrice Adhiambo also introduced the class of (BQ) operators in [12]. $\mathcal{G}$ is said to be (BQ) whenever $\mathcal{G}^{*2}\mathcal{G}^2$ commutes with $(\mathcal{G}^*\mathcal{G})$. It was shown in [12] that operator which is unitarily equivalent to a (BQ) operator is also in (BQ). A result was also given in [12] to show that any operator which is in (Q) is also in (BQ).

2. Preliminaries

Under this section, known definitions and results that will be useful in our main results are presented.

Definition 2.1. [8] $\mathcal{G} \in B(H)$ is n quasinormal if $\mathcal{G}^n\mathcal{G}^*\mathcal{G} - \mathcal{G}^*\mathcal{G}\mathcal{G}^n = 0$.

Definition 2.2. [8] $\mathcal{G} \in B(H)$ posses Bishop's property if all sequences of a function $f_a : U \longrightarrow H$ which is analytic and every open subset $\mathfrak{D}$ of $\mathbb{C}$ we have ; $(\zeta - \mathcal{G})f_a\zeta \longrightarrow 0$ as $a \longrightarrow \infty$ uniformly on all compact subset of $\mathfrak{D}$ and $f_a\zeta \longrightarrow 0$ as $a \longrightarrow \infty$ locally uniformly on $\mathfrak{D}$.

Definition 2.3. *An operator $\mathcal{G} \in B(H)$ is:*

(1). (Q) operator if $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{G}^*\mathcal{G})^2$.

(2). Unitary if $\mathcal{G}^*\mathcal{G} = \mathcal{G}\mathcal{G}^* = I$.

(3). (α, β)-Class (Q) operator provided $\alpha^2\mathcal{G}^{*2}\mathcal{G}^2 \leq (\mathcal{G}^*\mathcal{G})^2 \leq \beta^2\mathcal{G}^{*2}\mathcal{G}^2$.

(4). (α, β)-n-Class (Q) operator if $\alpha^2\mathcal{G}^{*2}\mathcal{G}^{2n} \leq (\mathcal{G}^*\mathcal{G}^n)^2 \leq \beta^2\mathcal{G}^{*2}\mathcal{G}^{2n}$ of 2-perinormal operators.

(5). (n, m)-normal if $\mathcal{G}^{*m}\mathcal{G}^n = \mathcal{G}^n\mathcal{G}^{*m}$ [2].

(6). (n, p)-quasinormal operator if $\mathcal{G}^n(\mathcal{G}^*\mathcal{G})^p = (\mathcal{G}^*\mathcal{G})^p\mathcal{G}^n$.

(7). Self-adjoint when $\mathcal{G}^* = \mathcal{G}$.

(8). A projection when $\mathcal{G}^* = \mathcal{G}$ (idempotent) and $\mathcal{G}^2 = \mathrm{I}$.

(9). $\mathcal{G}$ is said to be α-operator when $\mathcal{G}^3 = \mathcal{G}^*$ [5].

(10). $\mathcal{G}$ belongs to class K^* Quasi-n-class (Q) operator if $(\mathcal{G}^*)^k\mathcal{G}^{*2}\mathcal{G}^{2n} = (\mathcal{G}^*\mathcal{G}^n)^2(\mathcal{G}^*)^k$.

(11). $\mathcal{G}$ belongs to (BQ) if $\mathcal{G}^{*2}\mathcal{G}^2$ commutes with $(\mathcal{G}^*\mathcal{G})^2$.

(12). $\mathcal{G}$ belongs to Quasi-class (Q) if $\mathcal{G}\mathcal{G}^2\mathcal{G}^{*2} = (\mathcal{G}^*\mathcal{G})^2\mathcal{G}$ [10].

Definition 2.4. *$\mathcal{G} \in B(H)$ is isoloid whenever each point that is isolated belongs to $\sigma(\mathcal{G})$ is a member of $\sigma_p(\mathcal{G})$. S is polaroid provided all isolated points of $\sigma(\mathcal{G})$ are a pole of resolvent.*

Theorem 2.5. [3] $\mathcal{G}$ has Bishop's property whenever its n-power normal operator.

Theorem 2.6. [7] Let $\mathcal{G} \in B(H)$ be n-normal. Then $\mathcal{G}$ is isoloid and polaroid.

3. Main Results

Definition 3.1. *An operator $\bigtriangledown \in B(H)$ is called skew quasi-p-class (Q) if $(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$. We shall denote this class as $[V]$.*

Theorem 3.2. *Let $\bigtriangledown \in [V]$, then so are;*

(1). $\psi\triangle$ for any $\psi \in \mathbb{R}$.

(2). Every $\mathcal{N} \in B(H)$ unitarily equivalent to $\bigtriangledown$.

Proof.

(1). Suppose $\bigtriangledown \in [V]$, then;

$$\begin{aligned}
(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*) &= (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2 \\
&= ((\psi\bigtriangledown)^{*2}(\psi\bigtriangledown)^2)(\psi\bigtriangledown + (\psi\bigtriangledown)^*) \\
&= \overline{\psi}^2\bigtriangledown^{*2}\psi^2\bigtriangledown^2(\psi\bigtriangledown + \overline{\psi}\bigtriangledown^*) \\
&= \psi^8(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*)
\end{aligned}$$

and;

$$\begin{aligned}
&= (\psi\bigtriangledown + (\psi\bigtriangledown)^*)((\psi\bigtriangledown)^*\psi\bigtriangledown)^2 \\
&= (\psi\bigtriangledown + \overline{\psi}\bigtriangledown^*)\psi^2\overline{\psi}^2(\bigtriangledown^*\bigtriangledown)^2 \\
&= \psi^8(\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2.
\end{aligned}$$

Hence from ω and **??** $\psi\bigtriangledown$ is skew quasi-p-class (Q).

(2). Suppose $\mathcal{N} \in B(H)$ is unitarily equivalent to $\bigtriangledown$, $\exists$ unitary operator $\mathcal{U} \in B(H)$ such that $\bigtriangledown = \mathcal{U}^*\mathcal{N}\mathcal{U}$. Then;

$$\begin{aligned}
(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*) &= (\mathcal{U}^*\mathcal{N}\mathcal{U} + \mathcal{U}^*\mathcal{N}^*\mathcal{U})(\mathcal{U}^*\mathcal{N}^*\mathcal{U}\mathcal{U}^*\mathcal{N}^*\mathcal{U}\mathcal{U}^*\mathcal{N}\mathcal{U}\mathcal{U}^*\mathcal{N}\mathcal{U}) \\
&= (\mathcal{U}^*\mathcal{N}\mathcal{U} + \mathcal{U}^*\mathcal{Q}^*\mathcal{U})(\mathcal{U}^*\mathcal{N}^{*2}\mathcal{N}^2\mathcal{U}) \\
&= (\bigtriangledown\mathcal{U}^*\mathcal{U} + \bigtriangledown^*\mathcal{U}^*\mathcal{U})(\bigtriangledown^{*2}\mathcal{U}^*\mathcal{U}\bigtriangledown^2) \\
&= (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2.
\end{aligned}$$

Since, $(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$. Then, $(\mathcal{N}^{*2}\mathcal{N}^2)(\mathcal{N} + \mathcal{N}^*) = (\mathcal{N} + \mathcal{N}^*)(\mathcal{N}^*\mathcal{N})^2$.

□

Theorem 3.3. *Let $\bigtriangledown$ be a self-adjoint, then $\bigtriangledown \in [V]$.*

Proof.

$$(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown+\bigtriangledown^*) = (\bigtriangledown+\bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$$
$$(\bigtriangledown^2\bigtriangledown^2)(\bigtriangledown+\bigtriangledown) = 2\bigtriangledown^5 \qquad (\omega)$$
$$(\bigtriangledown+\bigtriangledown)(\bigtriangledown\bigtriangledown)^2 = 2\bigtriangledown^5 \qquad (\omega')$$

From ω and ω' $\bigtriangledown$ is skew quasi-p-class (Q) operator. ☐

Remark 3.4. *The example illustrates that the class of skew quasi-p-class (Q) does not preserve similarity of operators.*

Example 3.5. *Let* $\bigtriangledown$ *be an operator acting on a* $\mathbb{R}^2$ *such that* $\bigtriangledown = \begin{bmatrix} cce^{In(2)} & e^{i\pi}+1 \\ e^{i\pi}+1 & e^{2In(2)} \end{bmatrix}$ *and* $\mathcal{X} = \begin{bmatrix} e^{2In(2)} & e^{In(2)} \\ e^{In(2)} & e^{In(2)} \end{bmatrix}$. *It is easy to verify that;*

$$(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown+\bigtriangledown^*) = \begin{bmatrix} cce^{In(64)} & e^{i\pi}+1 \\ e^{i\pi}+1 & e^{In(2048)} \end{bmatrix} = (\bigtriangledown+\bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$$

Hence is skew quasi-p-class (Q). Now let; $\mathcal{X}\bigtriangledown\mathcal{X}^{-1} = \begin{bmatrix} cce^{i\pi}+1 & e^{In(16)} \\ e^{In(-8)} & e^{In(24)} \end{bmatrix} = \mathcal{M}$ *(say); it is easily seen that;*

$$(\mathcal{M}^{*2}\mathcal{M}^2)(\mathcal{M}+\mathcal{M}^*) = \begin{bmatrix} cce^{In(-491520)} & e^{In(-2719744)} \\ e^{In(1015808)} & e^{In(5603328)} \end{bmatrix}$$
$$(\mathcal{M}+\mathcal{M}^*)(\mathcal{M}^*\mathcal{M})^2 = \begin{bmatrix} cce^{In(-1376256)} & e^{In(5832704)} \\ e^{In(-7929856)} & e^{In(33619968)} \end{bmatrix}$$
$$\Rightarrow (\mathcal{M}^{*2}\mathcal{M}^2)(\mathcal{M}+\mathcal{M}^*) \neq (\mathcal{M}+\mathcal{M}^*)(\mathcal{M}^*\mathcal{M})^2$$

Hence does not preserve similarity.

Theorem 3.6. *If* $\bigtriangledown \in B(H)$, *then* $(\bigtriangledown^2\bigtriangledown^{*2})(\bigtriangledown^*+\bigtriangledown) = (\bigtriangledown^*+\bigtriangledown)(\bigtriangledown\bigtriangledown^*)^2$.

Proof. Since $\bigtriangledown \in [V]$, then by Theorem 3.3 so is $\bigtriangledown^*$; thus, $((\bigtriangledown^*)^{*2}(\bigtriangledown^*)^2)((\bigtriangledown^*)^*+\bigtriangledown^*) = (\bigtriangledown^*+(\bigtriangledown^*)^*)(\bigtriangledown^*)^*\bigtriangledown^*)^2$ implies that $(\bigtriangledown^2\bigtriangledown^{*2})(\bigtriangledown^*+\bigtriangledown) = (\bigtriangledown^*+\bigtriangledown)(\bigtriangledown\bigtriangledown^*)^2$. ☐

Theorem 3.7. *Let* $\bigtriangledown \in B(H)$, *it follows;*

(a). $\bigtriangledown+\bigtriangledown^* \in [V]$.

(b). $(\bigtriangledown^*\bigtriangledown)^2 \in [V]$.

(c). $\bigtriangledown^{*2}\bigtriangledown^2 \in [V]$.

(d). $I + \bigtriangledown^{*2}\bigtriangledown^{2}$, $I + (\bigtriangledown^{*}\bigtriangledown)^{2} \in [V]$.

Proof.

(a). Let $\mathcal{R} = \bigtriangledown + \bigtriangledown^{*}$ implies $\mathcal{R}^{*} = (\bigtriangledown + \bigtriangledown^{*})^{*} = \bigtriangledown^{*} + \bigtriangledown^{**} = \bigtriangledown^{*} + \bigtriangledown = \bigtriangledown + \bigtriangledown^{*} = \mathcal{R}$. $\mathcal{R}$ is self-adjoint and from Theorem 3.3, $\bigtriangledown \in [V]$.

(b). $(\bigtriangledown^{*}\bigtriangledown)^{2} = (\bigtriangledown^{*}\bigtriangledown)^{*2} = (\bigtriangledown^{**}\bigtriangledown^{*})^{2} = (\bigtriangledown\bigtriangledown^{*})^{2}$.

(c). $\bigtriangledown^{*2}\bigtriangledown^{2} = (\bigtriangledown^{*2}\bigtriangledown^{2})^{*} = \bigtriangledown^{**2}\bigtriangledown^{*2} = \bigtriangledown^{2}\bigtriangledown^{*2} = \bigtriangledown^{*2}\bigtriangledown^{2}$.

(d). $(I + \bigtriangledown^{*2}\bigtriangledown^{2}) = (I + \bigtriangledown^{*2}\bigtriangledown^{2})^{*} = I^{*} + \bigtriangledown^{**2}\bigtriangledown^{*2} = I + \bigtriangledown^{2}\bigtriangledown^{*2}$ and $I + (\bigtriangledown^{*}\bigtriangledown)^{2} = I^{*} + (\bigtriangledown^{*}\bigtriangledown)^{*2} = I + (\bigtriangledown\bigtriangledown^{*})^{2}$ and the proof for (b), (c) and (d) follows similarly from Theorem 3.3.

□

Theorem 3.8. *Let $\bigtriangledown \in B(H)$ be both self-adjoint and idempotent. If $\bigtriangledown \in [V]$, then its an (n, m)-normal operator.*

Proof. Suppose $\bigtriangledown\bigtriangledown \in [V]$;

$$\begin{aligned}
(\bigtriangledown^{*2}\bigtriangledown^{2})(\bigtriangledown + \bigtriangledown^{*}) &= (\bigtriangledown + \bigtriangledown^{*})(\bigtriangledown^{*}\bigtriangledown)^{2} \\
(\bigtriangledown^{*2}\bigtriangledown^{2})(\bigtriangledown + \bigtriangledown^{*}) &= \bigtriangledown^{*2}\bigtriangledown^{2}\bigtriangledown + \bigtriangledown^{*2}\bigtriangledown^{2}\bigtriangledown^{*} \\
&= \bigtriangledown^{*2}\bigtriangledown^{2} + \bigtriangledown^{*2}\bigtriangledown\bigtriangledown^{*} \quad \text{(Since } \bigtriangledown \text{ is idempotent)} \\
&= \bigtriangledown^{*2}\bigtriangledown^{2} + \bigtriangledown^{*2}\bigtriangledown^{2} \quad \text{(Since } \bigtriangledown \text{ is self-adjoint)} \\
&= 2\bigtriangledown^{*2}\bigtriangledown^{2} \qquad \text{(Y)}
\end{aligned}$$

Similarly; $(\bigtriangledown + \bigtriangledown^{*})(\bigtriangledown^{*}\bigtriangledown)^{2}$ yields;

$$\begin{aligned}
&= (\bigtriangledown^{*}\bigtriangledown)^{2}\bigtriangledown + (\bigtriangledown^{*}\bigtriangledown)^{2}\bigtriangledown^{*} \\
&= \bigtriangledown^{*2}\bigtriangledown^{2}\bigtriangledown + \bigtriangledown^{*2}\bigtriangledown^{2}\bigtriangledown^{*} \\
&= \bigtriangledown^{*2}\bigtriangledown^{2} + \bigtriangledown^{*2}\bigtriangledown^{2} \quad \text{(Since } \bigtriangledown \text{ is idempotent and self-adjoint)} \\
&= \bigtriangledown^{5}\bigtriangledown^{*2} + \bigtriangledown^{2}\bigtriangledown^{*2} \\
&= 2\bigtriangledown^{2}\bigtriangledown^{*2} \qquad \text{(Y}^{'}\text{)}
\end{aligned}$$

From (Y) and (Y$^{'}$) we have;

$$= \bigtriangledown^{*2}\bigtriangledown^{2} = \bigtriangledown^{2}\bigtriangledown^{*2}$$

Thus $\bigtriangledown$ is an (n, m)-normal operator; specifically a $(2, 2)$-normal operator. □

Theorem 3.9. *If $\bigtriangledown \in B(H)$ is quasi-p-normal , then $\bigtriangledown \in [V]$.*

Proof. Suppose $\bigtriangledown$ is quasi-p-normal;

$$\bigtriangledown^*\bigtriangledown(\bigtriangledown+\bigtriangledown^*) = (\bigtriangledown+\bigtriangledown^*)\bigtriangledown^*\bigtriangledown \tag{τ}$$

pre-multiplying and post-multiplying (τ) by $\bigtriangledown^*\bigtriangledown$ we obtain;

$$(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown+\bigtriangledown^*) = (\bigtriangledown+\bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$$

Hence the proof. □

Theorem 3.10. *Let $\bigtriangledown \in B(H)$ be a Quasi-Isometry and an Isometry, then $\bigtriangledown \in [V]$.*

Proof. By definition $\bigtriangledown^{*2}\bigtriangledown^2 = \bigtriangledown^*\bigtriangledown = I$; then

$$(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown+\bigtriangledown^*) = \bigtriangledown^*\bigtriangledown(\bigtriangledown+\bigtriangledown^*) = I(\bigtriangledown+\bigtriangledown^*) = (\bigtriangledown+\bigtriangledown^*) \tag{ς}$$

and;

$$(\bigtriangledown+\bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2 = (\bigtriangledown+\bigtriangledown^*)(I)^2 = (\bigtriangledown+\bigtriangledown^*)I = (\bigtriangledown+\bigtriangledown^*) \tag{ς'}$$

(ς) and (ς') points to $\bigtriangledown \in [V]$. □

Theorem 3.11. *Let $\bigtriangledown \in [V]$. If $\bigtriangledown$ is a Quasi-Isometry, then its a θ-operator.*

Proof. By definition; $(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown+\bigtriangledown^*) = (\bigtriangledown+\bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$. Suppose $\bigtriangledown$ is a Quasi-Isometry, then; $\bigtriangledown^*\bigtriangledown(\bigtriangledown+\bigtriangledown^*) = (\bigtriangledown+\bigtriangledown^*)\bigtriangledown^*\bigtriangledown$ as required. □

Remark 3.12. *The following result establishes the relationship between quasi-p-normal and skew quasi-p-class (Q) operators.*

Theorem 3.13. *If $\bigtriangledown \in B(H)$ is quasi-p-normal operator, then its $[V]$.*

Proof. Let $\bigtriangledown$ be quasi-p-normal, then;

$$\begin{aligned}(\bigtriangledown+\bigtriangledown^*)(\bigtriangledown^*\bigtriangledown) &= (\bigtriangledown^*\bigtriangledown)(\bigtriangledown+\bigtriangledown^*)\\ \bigtriangledown\bigtriangledown^*\bigtriangledown+\bigtriangledown^{*2}\bigtriangledown &= \bigtriangledown^*\bigtriangledown^2+\bigtriangledown^{*2}\bigtriangledown\\ \bigtriangledown^*\bigtriangledown^2+\bigtriangledown^{*2}\bigtriangledown &= \bigtriangledown^*\bigtriangledown^2+\bigtriangledown^{*2}\bigtriangledown\end{aligned}$$

pre-multiplying $\bigtriangledown\bigtriangledown^*$ and post-multiplying $\bigtriangledown\bigtriangledown^*$ on both sides;

$$\begin{aligned}\bigtriangledown^{*2}\bigtriangledown^3+\bigtriangledown^{*3}\bigtriangledown^3 &= \bigtriangledown^{*2}\bigtriangledown^3+\bigtriangledown^{*3}\bigtriangledown^3\\ (\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown+\bigtriangledown^*) &= (\bigtriangledown+\bigtriangledown\bigtriangledown^*)(\bigtriangledown^{*2}\bigtriangledown^2).\end{aligned}$$

Hence $\bigtriangledown \in [V]$. □

Definition 3.14. *$\bigtriangledown \in B(H)$ is skew quasi-n-p-class (Q) if $(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown^n + \bigtriangledown^{*n}) = (\bigtriangledown^n + \bigtriangledown^{*n})(\bigtriangledown^*\bigtriangledown)^2$, where n is a positive integer.*

Theorem 3.15. *Let $\bigtriangledown \in B(H)$. If ξ commutes with $\mathcal{C}$ and ζ commutes with $\mathcal{D}$ and $\xi^2\bigtriangledown = \xi^2\bigtriangledown$, then $\bigtriangledown$ is skew quasi-p-class (Q), where, $\xi^2 = (\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*)$, $\zeta^2 = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^{*2}\bigtriangledown^2)$, $\mathcal{C} = Re(\bigtriangledown) = \frac{\bigtriangledown + \bigtriangledown^*}{2}$ and $\mathcal{D} = Im(\bigtriangledown) = \frac{\bigtriangledown - \bigtriangledown^*}{2i}$.*

Proof. Since $\xi\mathcal{C} = \mathcal{C}\xi$, $\zeta\mathcal{D} = \mathcal{D}\zeta$. Then $\xi^2\mathcal{C} = \mathcal{C}\xi^2$ and $\zeta^2\mathcal{D} = \mathcal{D}\zeta^2$, so;

$$\xi^2\bigtriangledown + \xi^2\bigtriangledown^* = \bigtriangledown\xi^2 + \bigtriangledown^*\xi^2$$
$$\xi^2\bigtriangledown - \xi^2\bigtriangledown^* = \bigtriangledown\xi^2 - \xi^2\bigtriangledown^*$$
$$\bigtriangledown\xi^2 = \xi^2\bigtriangledown$$
$$\bigtriangledown(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown + \bigtriangledown^*) = \bigtriangledown(\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$$
$$(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown^2 + \bigtriangledown^{*2}) = (\bigtriangledown^2 + \bigtriangledown^{*2})(\bigtriangledown^*\bigtriangledown)^2$$

Similarly with $\zeta^2\mathcal{D} = \mathcal{D}\zeta^2$, we get;

$$\zeta^2\bigtriangledown - \zeta^2\bigtriangledown^* = \bigtriangledown\zeta^2 - \bigtriangledown^*\zeta^2$$
$$\bigtriangledown\zeta^2 = \zeta^2\bigtriangledown$$
$$\bigtriangledown(\bigtriangledown^*\bigtriangledown)^2(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2\bigtriangledown$$
$$(\bigtriangledown^*\bigtriangledown)^2(\bigtriangledown^2 + \bigtriangledown^{*2}) = (\bigtriangledown^2 + \bigtriangledown^{*2})(\bigtriangledown^{*2}\bigtriangledown^2)$$
$$(\bigtriangledown^{*2}\bigtriangledown^2)(\bigtriangledown^2 + \bigtriangledown^{*2}) = (\bigtriangledown^2 + \bigtriangledown^{*2})(\bigtriangledown^*\bigtriangledown)^2$$

Hence $\bigtriangledown$ is skew quasi-2-p-class (Q) operator. □

Theorem 3.16. *Let $\bigtriangledown_1$, $\bigtriangledown_2 \in B(H)$ be skew quasi-p-class (Q) operators such that $\bigtriangledown_1^{*2}\bigtriangledown_2^2 = \bigtriangledown_2^{*2}\bigtriangledown_1^2 = \bigtriangledown_1^{*2}\bigtriangledown_1 = \bigtriangledown_2^{*2}\bigtriangledown_1 = \bigtriangledown_2^{*2}\bigtriangledown_1^* = 0$, then $\bigtriangledown_1 + \bigtriangledown_2$ is skew quasi-p-class (Q) operator.*

Proof. By assumption, $\bigtriangledown_1$ and $\bigtriangledown_2$ are skew quasi-p-class (Q) operators, then;

$$= ((\bigtriangledown_1 + \bigtriangledown_2)^{*2}(\bigtriangledown_1 + \bigtriangledown_2)^2)((\bigtriangledown_1 + \bigtriangledown_2) + (\bigtriangledown_1 + \bigtriangledown_2)^*)$$
$$= ((\bigtriangledown_1^{*2} + \bigtriangledown_2^{*2})(\bigtriangledown_1^2 + \bigtriangledown_2^2))((\bigtriangledown_1 + \bigtriangledown_2) + (\bigtriangledown_1^* + \bigtriangledown_2^*))$$
$$= (\bigtriangledown_1^{*2} + \bigtriangledown_2^{*2})(\bigtriangledown_1^2 + \bigtriangledown_2^2))(\bigtriangledown_1 + \bigtriangledown_2 + \bigtriangledown_1^* + \bigtriangledown_2^*)$$
$$= (\bigtriangledown_1^{*2}\bigtriangledown_1^2 + (\bigtriangledown_1^{*2}\bigtriangledown_2^2 + \bigtriangledown_2^{*2}\bigtriangledown_1^2 + \bigtriangledown_2^{*2}\bigtriangledown_2)(\bigtriangledown_1 + \bigtriangledown_2 + \bigtriangledown_1^* + \bigtriangledown_2^*)$$

Since $\bigtriangledown_1^{*2}\bigtriangledown_2^2 = \bigtriangledown_2^{*2}\bigtriangledown_1^2 = 0$

$$= (\bigtriangledown_1^{*2}\bigtriangledown_1^2 + \bigtriangledown_2^{*2}\bigtriangledown_2)(\bigtriangledown_1 + \bigtriangledown_2 + \bigtriangledown_1^* + \bigtriangledown_2^*)$$

$$= \triangledown_1^{*2}\triangledown_1^2(\triangledown_1 + \triangledown_1^{*2}\triangledown_1^2(\triangledown_2 + \triangledown_1^{*2}\triangledown_1^2\triangledown_1^*$$
$$+ \triangledown_1^{*2}\triangledown_1^2\triangledown_2^* + \triangledown_2^{*2}\triangledown_2^2 + \triangledown_2^{*2}\triangledown_2^2\triangledown_1 + \triangledown_2^{*2}\triangledown_2^2\triangledown_2\triangledown_1^* + \triangledown_2^{*2}\triangledown_2^2\triangledown_2^*$$

Since $\triangledown_1^{*2}\triangledown_1 = \triangledown_2^{*2}\triangledown_1 = \triangledown_2^{*2}\triangledown_1^* = 0$;

$$\begin{aligned}
&= \triangledown_1^{*2}\triangledown_2^2\triangledown_1 + \triangledown_1^{*2}\triangledown_2^2\triangledown_2 + \triangledown_1^{*2}\triangledown_2^2\triangledown_1^* + \triangledown_1^{*2}\triangledown_2^2\triangledown_2^* \\
&= (\triangledown_2^2\triangledown_1 + \triangledown_2^2\triangledown_2 + \triangledown_2^2\triangledown_1^* + \triangledown_2^2\triangledown_2^*)\triangledown_1^{*2} \\
&= (\triangledown_1 + \triangledown_2 + \triangledown_1^* + \triangledown_2^*)(\triangledown_1^{*2}\triangledown_2^2) \\
&= ((\triangledown_1 + \triangledown_2) + (\triangledown_1 + \triangledown_2)^*)(\triangledown_1^*\triangledown_2)^2
\end{aligned}$$

Hence $\triangledown_1 + \triangledown_2$ is skew quasi-p-class (Q) operator. □

Theorem 3.17. *Let $\triangledown_1, \triangledown_2 \in B(H)$ be skew quasi-p-class (Q) operators such that $\triangledown_1^{*2}\triangledown_2^2 = \triangledown_2^{*2}\triangledown_1^2 = \triangledown_1^{*2}\triangledown_1 = \triangledown_2^{*2}\triangledown_1 = \triangledown_2^{*2}\triangledown_1^* = 0$, then $\triangledown_1 - \triangledown_2$ is skew quasi-p-class (Q) operator.*

Proof. The proof follows directly from Theorem 3.16. □

Theorem 3.18. *Let $\triangledown = \mathcal{U}|\triangledown|$ be the polar decomposition of $\triangledown \in B(H)$, then $\triangledown \in [V]$ if $|\triangledown|\mathcal{U} = \mathcal{U}|\triangledown|$.*

Proof. Let $\triangledown \in [V]$, then,

$$\begin{aligned}
&(\triangledown^{*2}\triangledown^2)(\triangledown + \triangledown^*) - (\triangledown + \triangledown^*)(\triangledown^*\triangledown)^2 \\
&\quad = ((\mathcal{U}^*|\triangledown|^2\mathcal{U}|\triangledown|^2)(\mathcal{U}|\triangledown| + \mathcal{U}^*|\triangledown|)) - ((\mathcal{U}|\triangledown| + \mathcal{U}^*|\triangledown|)(\mathcal{U}^*|\triangledown|\mathcal{U}|\triangledown|)^2) \\
&\quad = (|\triangledown|^2(\mathcal{U}^*\mathcal{U}|\triangledown|^2)(\mathcal{U}|\triangledown| + \mathcal{U}^*|\triangledown|)) - ((\mathcal{U}|\triangledown| + \mathcal{U}^*|\triangledown|)(|\triangledown|\mathcal{U}^*\mathcal{U}|\triangledown|)^2) \\
&\quad = (|\triangledown|^4((\mathcal{U}|\triangledown| + \mathcal{U}^*|\triangledown|)) - ((\mathcal{U}|\triangledown| + \mathcal{U}^*|\triangledown|)|\triangledown|^4) \\
&\quad = \mathcal{U}|\triangledown|^5 + \mathcal{U}^*|\triangledown|^5 - \mathcal{U}|\triangledown|^5 - \mathcal{U}^*|\triangledown|^5 = 0.
\end{aligned}$$

Hence, we get; $(\triangledown^{*2}\triangledown^2)(\triangledown + \triangledown^*) = (\triangledown + \triangledown^*)(\triangledown^*\triangledown)^2$ and thus $\triangledown \in [V]$. □

Theorem 3.19. *Let $\triangledown \in [V]$. If $\triangledown$ is both a quasi-isometry and an orthogonal projection, then its an n-normal operator.*

Proof. Since $\triangledown \in [V]$, we have; $(\triangledown^{*2}\triangledown^2)(\triangledown + \triangledown^*) = (\triangledown + \triangledown^*)(\triangledown^*\triangledown)^2$ by Theorem 3.21 we have; $\triangledown^{*2}\triangledown^2(\triangledown + \triangledown^*) = (\triangledown + \triangledown^*)\triangledown^{*2}\triangledown^2$. Suppose $\triangledown$ is a quasi-isometry, then; $\triangledown^*\triangledown(\triangledown + \triangledown^*) = (\triangledown + \triangledown^*)\triangledown^*\triangledown$. Similarly, if $\triangledown$ is an orthogonal projection then we have;

$$\triangledown^*\triangledown(\triangledown^2 + \triangledown^2) = (\triangledown^2 + \triangledown^2)\triangledown^*\triangledown$$
$$\triangledown^*\triangledown^3 = \triangledown^3\triangledown^*$$

Hence $\triangledown$ is an n-normal operator for $n = 3$. □

Theorem 3.20. *Let $\bigtriangledown \in B(H)$ be $\bigtriangledown \in [V]$. If $\bigtriangledown$ is both 2-self-adjoint and self-adjoint it follows its an n-quasinormal operator.*

Proof. By definition; $\bigtriangledown^{*2}\bigtriangledown^2(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$. Now suppose $\bigtriangledown$ is both 2-self adjoint and self adjoint, then; $\bigtriangledown^{*2} = \bigtriangledown^2 = \bigtriangledown^* = \bigtriangledown$; hence;

$$(\bigtriangledown^*\bigtriangledown)\bigtriangledown^2 = \bigtriangledown^2(\bigtriangledown^{*2}\bigtriangledown^2)$$
$$(\bigtriangledown^*\bigtriangledown)\bigtriangledown^2 = \bigtriangledown^2(\bigtriangledown^*\bigtriangledown)$$
$$\bigtriangledown^2(\bigtriangledown^*\bigtriangledown) = (\bigtriangledown^*\bigtriangledown)\bigtriangledown^2$$

Hence $\bigtriangledown$ is an n-quasinormal operator for $n = 2$. □

Theorem 3.21. *Let $\bigtriangledown \in B(H)$ be $\bigtriangledown \in [V]$, then $\bigtriangledown$ is a class (Q) if its unitary.*

Proof. $\bigtriangledown$ being skew quasi-p-class (Q), implies;

$$\bigtriangledown^{*2}\bigtriangledown^2(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2 \tag{ℸ}$$

pre-multiplying and post-multiplying (ℸ) by $\bigtriangledown^*$ and $\bigtriangledown$ respectively we have; $\bigtriangledown^{*2}\bigtriangledown^2(\bigtriangledown^*\bigtriangledown + \bigtriangledown\bigtriangledown^*) = (\bigtriangledown^*\bigtriangledown + \bigtriangledown^*\bigtriangledown)(\bigtriangledown^*\bigtriangledown)^2 \Rightarrow \bigtriangledown^{*2}\bigtriangledown^2 = (\bigtriangledown^*\bigtriangledown)^2$ as desired. □

Theorem 3.22. *If $\bigtriangledown \in [V]$ it follows it is isoloid and polaroid.*

Proof. The proof follows from Theorem 3.19 and Theorem 2.6 respectively. □

Theorem 3.23. *Let $\bigtriangledown \in B(H)$ be a self-adjoint skew quasi-p-class (Q) operator. Then $\bigtriangledown$ is an (n, p)-quasinormal operator.*

Proof. By definition; $\bigtriangledown^{*2}\bigtriangledown^2(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2$. Since $\bigtriangledown$ is class (Q) by Theorem 3.21 above, $(\bigtriangledown^*\bigtriangledown)^2(\bigtriangledown + \bigtriangledown^*) = (\bigtriangledown + \bigtriangledown^*)(\bigtriangledown^*\bigtriangledown)^2\bigtriangledown$ being self-adjoint ensures, $\bigtriangledown(\bigtriangledown^*\bigtriangledown)^2 = (\bigtriangledown^*\bigtriangledown)^2\bigtriangledown$. Hence $\bigtriangledown$ is (n, p)-quasinormal operator for $n = 1$ and $p = 2$. □

Theorem 3.24. *If $\bigtriangledown \in [V]$ then it has Bishop's property (property β).*

Proof. By Theorem 3.19 above $\bigtriangledown$ is an n-normal operator and since by Theorem 2.5 above, the proof follows for $\bigtriangledown$. □

4. Conclusion

The class of skew quasi-p-class (Q) has been introduced, it has been shown that this class enjoys relation with various classes such as n-normal, class (Q) and n-quasinormal among others. However, it has not been established whether this class enjoys existence of inequalities and we therefore recommend further research to be done to establish inequalities in this class.

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