MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS
2018/2019 ACADEMIC YEAR
THIRD YEAR TWO SEMESTER

SCHOOL OF SCIENCE
Bsc. MATHEMATICS

COURSE CODE: MAT 411
COURSE TITLE: FIELD THEORY

DATE: 26TH APRIL 2019 **TIME: 1430 - 1630 HRS**

INSTRUCTIONS TO CANDIDATES

1. Answer Question **ONE** and any other **TWO** questions.
2. All Examination Rules Apply.

<u>Question 1</u> [30 marks]

1 (a). State the meaning and give an example of:

i. a field of characteristic zero.
ii. an infinite field of characteristic p.
iii. an algebraic extension of a field F. [6 marks]

1 (b). Determine the vector space dimension of the field $\mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$ and exhibit three different bases. [6 marks]

1 (c). Prove that $\mathbb{Q}\left(\sqrt{5}, \sqrt{11}\right) = \mathbb{Q}(\sqrt{5} + \sqrt{11})$ [4 marks]

1 (d). Express $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2}$ in terms of elementary symmetric functions in x_1, x_2, x_3. [4 marks]

1 (e). Determine the splitting field of:

i. The polynomial $x^2 + x + 1$ over $\mathbb{Q}$.
ii. The polynomial $x^2 + x + 1$ over $\mathbb{Z}$. [5 marks]

1 (f). Give the definition of a solvable group and state its significance in Field Theory and solution of polynomial equations. [5 marks]

<u>Question 2</u> [20 marks]

E, F and K are three fields such that F is a finite extension of E and K is a finite extension of F.

i. Prove that

$$[K:E] = [F:E][K:F].$$

ii. Illustrate the result in (i) with $E = \mathbb{Q}$ and F and K specified with $[F:E] = 3$ and $[K:F] = 2$.

iii. Deduce from (i) that if a and b are algebraic over E of degree m and n respectively, then a/b is algebraic over E of degree $\leq mn$.

<u>Question 3</u> [20 marks]

3 (a). Factorize

$$x^9 - 1 \text{ in } \mathbb{Q}[x]$$

Hence, or otherwise determine the splitting field of

$x^6 + x^3 + 1$ over $\mathbb{Q}$.

3 (b). Let β denote the complex number $(\sqrt[4]{2} + i)^{-1}$.

i. Determine the minimal polynomial of β over $\mathbb{Q}$.

ii. Prove that $\mathbb{Q}(\sqrt[4]{2}, i) = \mathbb{Q}(\beta)$.

iii. Write down two other elements a and b in $\mathbb{Q}(\sqrt[4]{2}, i)$ such that

$$\mathbb{Q}(\beta) = \mathbb{Q}(a) = \mathbb{Q}(b).$$

<u>Question 4</u> [20 marks]

4 (a). Show that the polynomial $x^3 + x + 1$ is irreducible over $\mathbb{Z}_5$.

(b). Show that every element of the field $\frac{\mathbb{Z}_5[x]}{(x^3+x+1)\mathbb{Z}_5[x]}$ is of the form

$$r_0 + r_1 x + r_2 x^2 + M \text{ where } M = (x^3 + x + 1)\mathbb{Z}_5[x].$$

(C). Let α be a root of $x^3 + x + 1$, now considered as a polynomial in $\mathbb{Q}[x]$.

Express $\frac{1}{\alpha}$ and $\frac{1}{\alpha^2}$ as polynomials in α.

<u>Question 5</u> [20 marks]

Let K be an extension of F where K and F are fields and let $G(K, F)$ denote the group of automorphisms of K that leave F fixed elementwise.

a. Prove that
   i. $G(K, F)$ is a subgroup of the group of all automorphisms of K.
   ii. $o\big(G(K, F)\big) \leq [K: F]$
b. Give an example where the inequality in a(ii) is strict and an example where the equality holds.

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