MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS
2018/2019 ACADEMIC YEAR
THIRD YEAR TWO SEMESTER

SCHOOL OF SCIENCE
Bsc. MATHEMATICS

COURSE CODE: MAT 411
COURSE TITLE: FIELD THEORY

DATE: 26TH APRIL 2019 **TIME: 1430 - 1630 HRS**

INSTRUCTIONS TO CANDIDATES

1. Answer Question **ONE** and any other **TWO** questions.
2. All Examination Rules Apply.

<u>Question 1</u> [30 marks]

1 (a). State the meaning and give an example of:

i. a field of characteristic zero.
ii. an infinite field of characteristic p .
iii. an algebraic extension of a field F . [6 marks]

1 (b). Determine the vector space dimension of the field $Q(\sqrt{2}, i)$ over Q and exhibit three different bases. [6 marks]

1 (c). Prove that $Q(\sqrt{5}, \sqrt{11}) = Q(\sqrt{5} + \sqrt{11})$ [4 marks]

1 (d). Express $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2}$ in terms of elementary symmetric functions in x_1, x_2, x_3 . [4 marks]

1 (e). Determine the splitting field of:

i. The polynomial $x^2 + x + 1$ over Q .
ii. The polynomial $x^2 + x + 1$ over Z . [5 marks]

1 (f). Give the definition of a solvable group and state its significance in Field Theory and solution of polynomial equations. [5 marks]

<u>Question 2</u> [20 marks]

E, F and K are three fields such that F is a finite extension of E and K is a finite extension of F.

i. Prove that

$$[K:E]=[F:E][K:F].$$

ii. Illustrate the result in (i) with $E=Q$ and F and K specified with $[F:E]=3$ and $[K:F]=2$.

iii. Deduce from (i) that if a and b are algebraic over E of degree m and n respectively, then $\frac{a}{b}$ is algebraic over E of degree $\leq mn$.

<u>Question 3</u> [20 marks]

3 (a). Factorize

$$x^9 - 1 \quad \text{in} \quad Q[x]$$

Hence, or otherwise determine the splitting field of $x^6 + x^3 + 1$ over Q.

3 (b). Let β denote the complex number $(\sqrt[4]{2} + i)^{-1}$.

i. Determine the minimal polynomial of β over Q.

ii. Prove that $Q(\sqrt[4]{2}, i) = Q(\beta)$.

iii. Write down two other elements a and b in $Q(\sqrt[4]{2}, i)$ such that

$$Q(\beta) = Q(a) = Q(b).$$

<u>Question 4</u> [20 marks]

4 (a). Show that the polynomial $x^3 + x + 1$ is irreducible over Z_5 .

(b). Show that every element of the field $\frac{Z_5[x]}{(x^3 + x + 1)Z_5[x]}$ is of the form

$r_0 + r_1 x + r_2 x^2 + M$ where $M = (x^3 + x + 1)Z_5[x]$.

(C). Let α be a root of $x^3 + x + 1$, now considered as a polynomial in $Q[x]$.

Express $\frac{1}{\alpha}$ and $\frac{1}{\alpha^2}$ as polynomials in α .

<u>Question 5</u> [20 marks]

Let K be an extension of F where K and F are fields and let $G(K, F)$ denote the group of automorphisms of K that leave F fixed elementwise.

a. Prove that
   i. $G(K, F)$ is a subgroup of the group of all automorphisms of K .
   ii. $o(G(K, F)) \leq [K : F]$
b. Give an example where the inequality in a(ii) is strict and an example where the equality holds.

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