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Analysis of a fishery model with a depensation term

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Abstract
We formulate a mathematical fishery model with over-fishing induced Allee effect and stock dependent harvesting, since fish populations when reduced to low densities experience reduced rates of survival and reproduction. The model takes the form of a set of differential equations in which the population growth equation $\dot{n} = rn\left(\frac{n}{T} - 1\right)\left(1 - \frac{n}{K}\right) - qnE,$ has a depensation term. The other two equations describe the evolution of the fishing mortality *E* and the market price *p*. Aggregation is performed to reduce the model from a system of three equations to a system of two equations. Analysis of the model yields four points of equilibria. Local stability analysis is done for the equilibria points with zero harvesting.

Keywords: Depensation, allee effect, fishing mortality

1. Introduction
In population dynamics, the growth of a population can be described if the functional behavior of the rate of growth is known. Single species population models have been considered in several forms, see for instance [2, 3, 4, 6, 12], involving ordinary differential equations, delay differential equations and the discrete maps. Most often, logistic equation is used with an assumption that reducing density results in the same or higher per capita growth which implies that populations are resilient and recover rapidly when factors causing the decline are removed.
Fish populations exhibit social co-operation behaviourism which provides individuals a greater chance of survival and reproduction as the density increases, whereas when reduced to low densities, they experience reduced survival and reproduction rates. Over-fishing can cause the population of a fishery to persist at low densities, this is attributed to a phenomena known as the Allee effect in population dynamics. Allee effect is defined as " a positive relationships between component of individual fitness and either numbers or density of conspecifics", Allee effect therefore produces a positive density dependence in which fitness increases with increasing density, see for instance [1,11,24].
Bio-economics mainly considers the exploitation of renewable resources from the point of control theory see for instance [10,13], focussing on the existence of the maximum sustainable yield (MSY), which is the maximum proportion that can be removed from the stock over time without causing population decline below the optimum level. This theory however is not necessarily the best management see for instance [15], since the long run consumption profile does not coincide with that of utility maximization. MSY policy only takes into consideration the benefits of resource exploitation but completely disregards the cost of operation of resource exploitation.

1.1 Earlier Fishery models
Fish population models have been considered in numerous works with interest in fishery management in consideration due to the ecological effects of harvesting, see for instance [10]. Many fishery models takes into account two state variables; *n* (*t*), the resource measured in Kg per unit area and *E* (*t*), fishing mortality. Earlier contributions had models which read as

$$\dot{n}(t) = f(n) - h(n,E)$$

$$\dot{E}(t) = \beta\,(ph(n,E) - cE), \qquad (1)$$

Where $f(n)$ and $h(n,E)$ are the growth and harvesting functions respectively, p the price of a unit resource c, the cost per unit of harvesting the resource and β an adjustment parameter. In [23] a time continuous model is considered while in [7] a time discrete version is considered. Mathematical models such as in Equation (1) consider two equations; the first equation represents the variation with time of the fishery resource density which is due to the natural growth and harvesting. The second equation describes the time variation of the number of firms involved in resource exploitation which is assumed to increase when the exploitation is rentable and conversely. Reference is also made to the fishery models in [21]. The fishery was assumed to grow logistically with $f(n)$ taking the form

$$f(n) = rn\left(1 - \frac{n}{K}\right),$$

With r the intrinsic growth rate and K, the carrying capacity in the first Equation of (1). Harvesting is according to Type 1 functional response commonly known as the Schaefer function where

$$H(n,E) = qnE,$$

With q, a positive constant called capturability or catchability. In [18], the same variables are considered but with an addition of the third equation describing the nonlinear variation in market price.
Several models have been also been considered of fish population with effects of over- fishing and with Allee effect,

$$\dot{y} = -ry\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right) - Ey \qquad (2)$$

For $y > 0$, $0 < T < K$, where r, T and K are the parameters of a given fish population whose population is denoted by $y(t)$ and $E(t)$ is the fishing effort is considered in [8] while

$$\dot{n} = n(n - a)(1 - n) - fn, \qquad (3)$$

where $0 < a < 1$ denote the Allee effect, $n(t)$ the fish population at time t and f the fishing rate, is considered in [12].
In this paper, we formulate a continuous time mathematical model of a fishery describing the evolution of the fish population that exhibits Allee effect, the fishing mortality and the market price. In the real world ecological habitats, the model can reflect a telescoping of effects which influence the population. Since, in a real ecological habitat, there is a complexity of time evolutions of the natural populations due to the nonlinearity of the biological growth functions and the ecological complexity of interactions among species accompanied with resource exploitation for economic benefits. A traditional, continuous time model of an open access considered in cases above, based on the logistic net growth, does not allow for extinction see for instance [11]. Existence of fish populations at low population levels hence a risk of extinction or even eventual extinction has been reported, see for instance [9, 16]. A growth function with a depensation term, is thus more plausible to examine the existence of such populations at this unstable population level states.

2. The model Equation

The study of the population dynamics at the unstable population level states is the main concern coupled with harvesting that is driven by the market price. The most feasible model to study this variables is thus:

$$\dot{n} = f(n) - h(n,E)$$

$$\dot{E} = \beta(\rho h(n,E) - cE)$$

$$\dot{p} = ap(D(p) - S(p)), \qquad (4)$$

With the variables and parameters having the same meaning and sense as in [18], though with the equation in population having a different derivation.
If $n(t)$ is the population of the fish species at the time t, then the rate of change of the population
$\dot{n}$ = birth − death + Migration
Is the conservation equation for the population. If we consider the simplest form of the conservation equation, then we have

$$\dot{n} = bn(t) - dn(t), \qquad (5)$$

Equivalent to

$$\dot{n} = rn. \qquad (6)$$

Where $r = b-d$ is a constant intrinsic growth rate. Such a population growth exhibited by (6), may be valid within a very short period of time but cannot persist forever, since r changes with $n(t)$ due to population limiting factors like food supply and oxygen levels occasioned by intraspecific interactions. If $r := r(n)$ is now considered to be dependent on $n(t)$, such that $r(n)$ is a decreasing linear function of $n(t)$ supposedly given by

$$R(n) = r - an. \tag{7}$$

When equation (7) is brought into equation (6), we obtain

$$\dot{n} = rn\left(1 - \frac{n}{k}\right), \tag{8}$$

where K is the limiting population referred to as the carrying capacity. When $n > K$, $\dot{n} < 0$ and the population decreases since the members of the species compete with each other for limited resources whereas if $n < K$, $\dot{n} > 0$ and the population grows approaching K. Since existence of fish population at low levels is a phenomena in fisheries and even extinction of species a reality in several ecological habitats, a change of the sign of r to negative and replacing the carrying capacity K with a threshold population T gives us the equation;

$$\dot{n} = -rn\left(1 - \frac{n}{T}\right), \tag{9}$$

In this equation, if $n(t) < T$, the population dies out whilst when $n(t) > T$, the population explodes. The decay of the population explains the failure of the species to benefit from social cooperative behaviourism like enhanced reproduction and predator surveillance. The rapid growth in population is occasioned by positive population density factors where the species enjoys high reproduction rates and enhanced predator surveillance but since the population cannot explode uncontrollably, we consider a model equation with both threshold population T and the carrying capacity K. An equation to describe such growth dynamics is

$$\dot{n} = rn\left(1 - \frac{n}{K}\right)\left(\frac{n}{T} - 1\right), \tag{10}$$

Taking into account the fishing mortality, we represent this with the equation $h(n, E) = qnE$ where q is a catchability constant and nE is the fish biomass harvested due to the fishing effort E see for instance [22]. The complete equation with a depensation term now takes the form

$$\dot{n} = rn\left(1 - \frac{n}{K}\right)\left(\frac{n}{T} - 1\right) - qnE, \tag{11}$$

The Equation (11) together with the equation in harvesting and in market price as seen in [18] constitutes the model which reads;

$$\begin{aligned} \dot{n} &= rn\left(\frac{n}{T} - 1\right)\left(1 - \frac{n}{K}\right) - qnE, \\ \dot{E} &= \beta E(pqn - c), \\ \dot{p} &= ap(A - p - qnE), \end{aligned} \tag{12}$$

3. Equilibrium points

3.1 Aggregation

The model in Equation (12) can be reduced to a model of two equations through aggregation whereby We consider that the market price evolves comparatively faster than the fish stock and the fishing effort since suppliers adjust to the market prices and the fishery conditions in order to avoid loses and recoup their investment and make profit. Appropriate aggregation enables the reduction of the model to a system of two differential equations, See for instance [22] and [24]. The price in the harvesting effort is replaced with the nontrivial equilibrium values $p := p^*$. p^*, which solves

$$\dot{p} = \alpha p(A - p - qnE) = 0 \tag{13}$$

To obtain

$$p^* = A - qnE. \tag{14}$$

Such that the second equation now becomes

$$E^{\cdot} = \beta E\,((A - qn\,E)\,qn - c) \tag{15}$$

We take $\beta = 1$, which is a maximum value in the range $0 \le \beta \le 1$ and may occur when the environmental conditions and harvesting are favorable for stock growth in the fishery. This aggregation makes us obtain;

$$\begin{aligned} \dot{n} &= rn\left(\frac{n}{T}-1\right)\left(1-\frac{n}{K}\right) - qnE, \\ \dot{E} &= E\big((A - qnE)qn - c\big), \end{aligned} \tag{16}$$

A system of two differential equations.

3.2 Equilibrium points

Equilibrium points for the system in Equation (16) are the n-nullclines and the E-nullclines with

$$\begin{aligned} & rn\left(\frac{n}{T}-1\right)\left(1-\frac{n}{K}\right) - qnE = 0 \\ & E\big((A - qnE)qn - c\big) = 0, \end{aligned} \tag{17}$$

We have

$$\begin{aligned} (n_0, E_0) &= (0, 0) \\ (n_1, E_1) &= (T, 0) \\ (n_2, E_2) &= (K, 0) \\ (n_3, E_3) &= (n^*, E^*) \end{aligned} \tag{18}$$

Where (n^*, E^*) are the interior points that are solutions of the system

$$\begin{aligned} E(n) &= \frac{r}{q}\left(\frac{n}{T}-1\right)\left(1-\frac{n}{K}\right), \\ E(n) &= \frac{1}{qn}\left(A - \frac{c}{qn}\right) \end{aligned} \tag{19}$$

Whose equilibrium points can be denoted by E_1, E_2, E_3

3.3 Local stability analysis

To analyse the stability of the equilibrium points of (16), we linearise the system about the equilibrium points and study the trace and determinant of the matrix of linearisation, as various parameters are varied. The matrix of linearisation defines a linear map which is the best linear approximation of the function near the equilibrium point and it gives information about the local behaviour of the function. We consider the equilibrium points where the harvesting is zero.

Equation (16) can be expressed as

$$\begin{aligned} f(n,E) &:= nr\left(1-\frac{n}{k}\right)\left(\frac{n}{T}-1\right) - qnE, \\ g(n,E) &:= -cE + qnE(A - qnE), \end{aligned} \tag{20}$$

The Jacobian matrix is

$$\begin{aligned} J(n,E) &:= \begin{pmatrix} f_n(n,E) & f_E(n,E) \\ g_n(n,E) & g_E(n,E) \end{pmatrix} \\ &= \begin{pmatrix} n\left(\frac{2r}{T} - \frac{3rn^2}{Tk} + \frac{2r}{K}\right) - qE - r & -qn \\ qEA - 2q^2nE^2 & -c + qnE - 2q^2n^2E \end{pmatrix} \end{aligned}$$

At E0 the Jacobian matrix is

Whose eigenvalues are: −r and −c. Since both are negative, the equilibrium point E0 is a stable equilibrium point.
At E1,

$$J(k,0)=\begin{pmatrix} r\left(1-\frac{K}{T}\right) & -qk \\ 0 & -c+qkA \end{pmatrix}$$

Since $r\langle\langle K, r\langle\langle T$ and $K\rangle T$ then $r\left(1-\frac{K}{T}\right)<0$ if $K<\frac{c}{Aq}$, then $J(K,0)$ is a stable equilibrium point, for both eigenvalues are negative but if $K>\frac{c}{Aq}$, then $J(K,0)$ is a saddle point since one eigenvalue is negative and the other positive.
At the equilibrium E_2, the Jacobian matrix is given by

$$J(T,0)=\begin{pmatrix} r\left(1-\frac{T}{K}\right) & -qT \\ 0 & -c+qTA \end{pmatrix}$$

Since $T<K, r\left(1-\frac{T}{K}\right)>0$ if $T<\frac{c}{Aq}$ then $J(T, 0)$ is a saddle while if $K>\frac{c}{Aq}$, then $J(T,0)$ is unstable since both eigenvalues are positive.

4. Discussion and Conclusion

We have presented a fishery model with a depensation term which addresses the dynamics at the unstable threshold population levels T. Analysis of the model through aggregation and determination of equilibrium solutions yields four equilibrium points

$$(n_0, E_0) = (0, 0)$$
$$(n_1, E_1) = (T, 0)$$
$$(n_2, E_2) = (K, 0)$$
$$(n_3, E_3) = (n^*, E^*) \qquad (21)$$

It is observed that inclusion of the depensation term in the growth equation yields one more equilibrium point $(n_1, E_1) = (T, 0)$ not earlier obtained in [18]. Local stability analysis performed on the equilibria shows that $(n_0, E_0) = (0, 0)$ is a stable equilibrium point. Any population having a value close to zero will move to zero in time even in the absence of harvesting. This is trivial since the absence of fish population implies no harvesting can be done and this state is stable. The equilibrium $(n_1, E_1) = (T, 0)$ is a either a saddle or unstable depending on the values of the parameters c, A, q and T. Though in both cases, this equilibrium point is unstable. In the absence of harvesting population close to T will decay towards zero or grow logistically to the carrying capacity in case of other perturbation other than human harvesting. This is feasible since human harvesting is not the only cause mortality. The equilibrium $(n_2,E_2) = (K,0)$ is either a saddle or stable, in most ecological habitants, this equilibrium will be stable since in the absence of human harvesting, the populations beyond the carrying capacity decays towards carrying capacity whereas those populations below the carrying capacity but above the threshold population will grow towards the carrying capacity in time.
The equilibrium points here discussed poses minimal economic activity for the fishery but fisheries for recreational benefits like sport fishing may be addressed by these long term behaviour. The equilibrium $(n_3, E_3) = (n^*, E^*)$ is of most concern to both economists and conservationists since its analysis will give the behaviour real prediction of extinction or sustainable harvesting. This analysis shall be performed at varying values of of the threshold population T.

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