MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS
2016/2017 ACADEMIC YEAR
YEAR III SEMESTER I

SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES
BACHELOR OF SCIENCE

COURSE CODE: STA3124
COURSE TITLE: THEORY OF ESTIMATION

DATE: 7TH DEC,2018 **TIME: 11-1PM**

INSTRUCTIONS TO CANDIDATES

1. Answer Question **ONE** and any other TWO questions.
2. All Examination Rules Apply.

Question 1(30 Marks)

a) Define the following terms as used in Theory of Estimation.

i) Estimator (2mks)

ii) Estimate (2mks)

iii) Parameter (2mks)

iv) Statistic (2mks)

b) $X_1, X_2, ..., X_n$ is a random sample from the uniform distribution between θ and 1 i.e $f(x) = (1-\theta)^{-1}, \theta < x < 1$ where θ is an unknown parameter. Denote the sample mean by $\overline{X}$.

(i) Show that the method of moments estimator, $\hat{\theta}$, of θ is $2\overline{X} - 1$. (3mks)

(ii) Show that $\hat{\theta}$ is unbiased and find its variance. (4mks)

(iii) Let $Y = \min\{X_1, X_2, ..., X_{n1}\}$. By finding $P(Y > y)$, show that the pdf of Y is

$$f(y) = \frac{n(1-y)^{n-1}}{(1-\theta)^n}, \theta < y < 1$$ (4mks)

c) (i) Define what is meant by a sufficient statistic (2mks)

Let $X_1, X_2, ..., X_n$ be a random sample from

$f(x;\theta) = \theta^x (1-\theta)^{1-x}, x = 0,1 , \quad 0 < \theta < 1$

(ii) Show that $T = \sum X_i$ is a sufficient statistic for θ. (3mks)

d) The random variable X has an exponential distribution

$f(x \; ; \lambda) = \lambda e^{-\lambda x} \; , \; x > 0 \; ; \; \lambda > 0$

(i) If $X_1, X_2, ..., X_n$ is a random sample from this distribution, derive maximum likelihood estimator of λ. (3mks)

(ii) Derive expectation of the distribution (2mks)

(iii) Show that $\hat{\lambda}$ is an unbiased estimator of λ. (1mk)

Question 2(20 Marks)

(a) (i) Define what is meant by point estimation (2mks)

(ii) State three methods of finding point estimators (3mks)

Let $X_1, X_2, ..., X_n$ be a random sample from

$$f(x \; ; \; \theta \;) = (1+\theta)x^{\theta+1} \quad 0 < x < 1 \; ; \theta > 1$$

(iii) Find the method of moments estimator for θ. (3mks)

(iv) For the observed value.

0.25, 0.75, 0.45, 0.85, 0.55, 0.85, 0.95, 0.90, find the estimate in (iii). (2mks)

(b) (i) Define the term biased estimator (1mk)

Let $X_1, X_2, ..., X_n$ be a random sample from $U[0,\theta]$ where $\theta > 0$. You are given that the distribution has mean $\frac{\theta}{2}$ and variance $\frac{\theta^2}{12}$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and let $Y = \max X_i$.

Consider 3 estimators of $\hat{\theta}_1 = 2\bar{X}$, $\hat{\theta}_2 = Y$, $\hat{\theta}_3 = \left(\frac{n+1}{n}\right)Y$

Given that the probability density function of Y is

$$g(y) = \frac{ny^{n-1}}{\theta^n}, \qquad 0 < y < \theta$$

(ii) Find bias and variance of each of these estimators. (9mks)

<u>Question 3(20 Marks)</u>

a) The random variable X has moment generating function (mgf) $M_X(t)$. Let

$R(t) = \log M_X(t)$.

(i) Find $R'(t)$ and $R''(t)$ in terms of the first two derivatives of $M_X(t)$. (2mks)

(ii) Show that $R'(0) = E(X)$ and $R''(0) = Var(X)$ (4mks)

b) The random variable X has a Poisson distribution with pmf

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2,... \quad ;\lambda > 0$$

(i) Show that the mgf of X is $M_X(t) = e^{\lambda(e^t - 1)}$. (3mks)

(ii) Use the mgf to show that $E(X) = \lambda$.

Given also that

$E(X^2) = \lambda + \lambda^2$ and $E(X^3) = \lambda + 3\lambda^2 + \lambda^3$, show that

$E[(X - E(X))^3] = \lambda$ (5mks)

(iii) $X_1, X_2, ..., X_n$ are independent and identically distributed Poisson random variables each with pmf as given above. Use their mgfs to find the distribution of their sum $S = \sum_{i=1}^{n} X_i$

State any properties of mgfs which you use in your solution (6mks)

Question 4 (20 Marks)

The random sample $X_1, X_2, ..., X_n$ comes from a $N(\mu, \sigma^2)$ distribution.

(i) Show that

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{(n-1)\sigma^2}{n} \text{ where } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and $n > 1$. Use this result to obtain an unbiased estimator for σ^2.

(Assume $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \frac{\sigma^2}{n}$) (9mks)

A general estimator of the form $S_a{}^2 = (an - a - 1)\sigma^2$.

(ii) Show that the bias of $S_a{}^2$ is $(an - a - 1)\sigma^2$ (4mks)

(iii) The mean square error (MSE) of an estimator is defined to be the variance of the estimator plus the square of its bias. Given that

$Var\left(\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right) = 2(n-1)\sigma^4$, find the MSE of $S_a{}^2$ and show that the value of a which minimizes this MSE is $\frac{1}{n+1}$. (7mks)