UNIVERSITY EXAMINATIONS, 2018
FOURTH YEAR EXAMINATION
FOR
THE DEGREE OF BACHELOR OF MATHEMATICS
MAT 427- ORDINARY DIFFERENTIAL EQUATIONS II

Instructions to candidates:

Answer **Question 1** *and* **TWO other Questions**.

All Symbols have their usual meaning.

Remember to use correct English at all times.

DATE: *December* 2018 TIME:2hrs

Question 1(Entire course: 30 Marks)

(a) Let $A \in L(R^n)$. Show that the unique solution of the initial value problem

$$\dot{x} = Ax, \quad x(0) = K \in R^n \tag{1}$$

is $e^{tA}K$. **(4 Marks)**

(b) Find the solution $x(t, x_0)$ of the initial value problem

$$\dot{x} = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} x, \quad x(0) = (K_1, K_2, 0),$$

where K_1 and K_2 are nonzero constants . **(3 Marks)**
What is the value of

$$\lim_{t \to \infty} |x(t, x(0))|?$$

(1 Mark)

(c) Let the 2×2 matrix A have real, distinct eigenvalues λ, μ. Suppose that an eigenvector of λ is $(1, 0)^T$ and an eigenvector of μ is $(-1, 1)^T$. Sketch the phase portraits of $\dot{x} = Ax$ for the following cases and state the nature of stability :

(i) $0 < \lambda < \mu$, **(2 Marks)**

(ii) $0 < \mu < \lambda$, **(2 Marks)**

(iii) $\lambda < \mu < 0$, **(2 Marks)**

(iv) $\lambda < 0 < \mu$, **(2 Marks)**

(v) $\lambda = 0, \mu > 0$ **(2 Marks)**.

(d) Consider the equation

$$\dot{x} = (x - 2)(x + 1). \tag{2}$$

Sketch on the same diagram the possible solution curves for Equation (2) for the following initial conditions

- $-\infty < x(0) < -1$ **(1 Mark)**
- $x(0) = -1$ **(1 Mark)**
- $-1 < x(0) < 2$ **(1 Mark)**
- $x(0) = 2$ **(1 Mark)**
- $2 < x(0) < \infty$ **(1 Mark)**

(e) Consider a simple harmonic oscillator

$$m\ddot{x} + kx = 0, \tag{3}$$

where m is the mass at the end of an elastic spring, and x is the extension from its rest position.
Reduce the system into a first order system. (**1 Marks**)
Sketch its possible phase plane. (**2 Marks**)
The diagram below shows the relative position of a mass at the end of an elastic spring. The line marked $x = 0$ is the rest position.

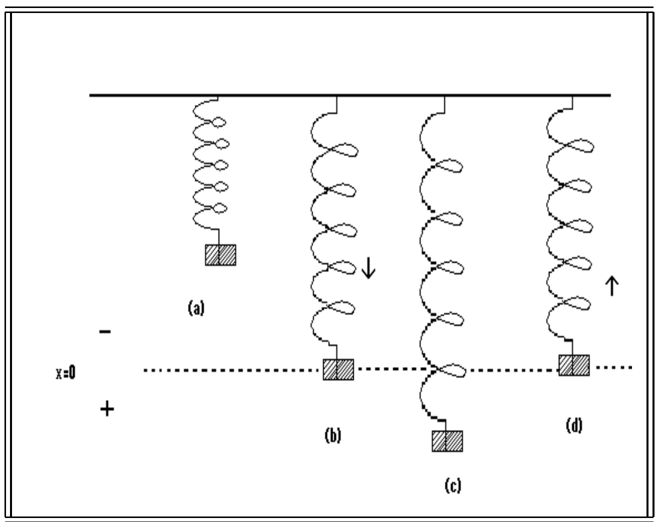

Figure 2 Oscillating mass attached to a spring.

The arrows indicate the direction of motion.
• Indicate the relative position of diagrams (a),(b), (c),(d) above in a phase plane to the system (3) (**4 Marks**)

Question 2 (Linear System: 20 Marks) Consider the nonhomogeneous initial value problem

$$\dot{x}(t) = Ax(t) + b(t), \quad x(0) = x_0, \tag{4}$$

where $A \in \mathbf{R}^{n\times n}$, $b(t) \in \mathbf{R}^n$ are continuous functions, and other symbols have their usual meaning.

(a) What do you understand by $X(t) \in \mathbf{R}^{n\times n}$ is a fundamental solution of the differential equation (4) with $b(t) = 0$. (**3 Marks**)

(b) Derive the *variation-of-constants formula* for Equation (4) above. (**6 Marks**)

(c) By using the method in (b) above, solve the system of equations below

$$\begin{aligned} \dot{x_1} &= x_1 + x_2 + t, \\ \dot{x_2} &= -x_2 + 1, \end{aligned} \tag{5}$$

subject to $x_1(0) = 1,\ x_2(0) = 0.$ **(11 Marks)**

Question 3 (General Properties of Differential Equations: 20 Marks)

(a) Consider the initial value problem

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{6}$$

where $f \in \mathcal{C}(D, \mathbf{R}) \quad D \subset I \times \mathbf{R}$. Prove that $x(t)$ is a solution of this initial value problem for $t \in I$ **if and only if** $x(t)$ is a continuous function that satisfies the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))ds, \qquad \text{for all } t \in I \tag{7}$$

(10 Marks)

(b) Let $\mathcal{B}(\mathbf{R})$ be a set of all bounded functions on $\mathbf{R}$. Show that for $\mu > 0$, $b \in \mathcal{B} \cap \mathcal{C}(\mathbf{R})$

$$x(t) := \int_{-\infty}^t \exp(-\mu(t-s))\, b(s)\, ds$$

is the only solution of

$$x'(t) = -\mu x + b(t), \qquad \text{in } \mathcal{B}(\mathbf{R}),$$

(10 Marks)

Question 4 (Modeling and Stability through Linearization: 20 Marks)

(a) Consider the differential equation $\dot{x} = Ax$, where $A \in R^{2\times 2}$ constant matrix. By using the determinant and trace of A, determine under what conditions will one expect to get a stable or unstable

(i) Nodes, **(2 Marks)**

(ii) Saddles, **(2 Marks)**

(iii) Centers, **(1 Mark)**

(iv) Foci at the origin. **(2 Marks)**

(b) Suppose two species X and Y are to be introduced onto an island. It is known that the two species compete, but the precise nature of their interactions is unknown. We assume that the populations $x(t)$ and $y(t)$ of X and Y, respectively, at time t are modeled by a system

$$\dot{x} = f(x, y), \tag{8}$$
$$\dot{y} = g(x, y). \tag{9}$$

In the questions below, justify your answers.

(i) Suppose $f(0,0) = g(0,0) = 0$; that is, $(x, y) = (0,0)$ is an equilibrium point. What does this say about the ability of X and Y to migrate to the island? (**2 Marks**)

(ii) Suppose that a small population of just X or just Y will rapidly reproduce. What does this imply about $f_x(0,0)$ and $g_y(0,0)$? (**2 Marks**)

(iii) Since X and Y compete for resources, the presence of either of the species will decrease the rate of growth of the population of the other. What does this say about $f_y(0,0)$ and $g_x(0,0)$? (**2 Marks**)

(iv) Using the assumption from parts (i) through (iii), what type(s) of equilibrium point (sink, source, center, and so on) could $(0,0)$ possibly be?[*Hint*: there may be more than one possibility; if so list them all.] (**2 Marks**)

Suppose that species X reproduces very quickly if it is on the island without any Y's present, and that species Y reproduces slowly if there are no X's present. Also suppose that the growth rate of species X is decreased a relatively large amount by the presence of Y, but that species Y is indifferent to X's population.

(v) What can you say about $f_x(0,0)$ and $g_y(0,0)$? (**2 Marks**)

(vi) What can you say about $f_y(0,0)$ and $g_x(0,0)$? (**2 Marks**)

(vii) What are the possible type(s)(sink, source, center, and so on) for the equilibrium point at $(0,0)$ [*Hint*: there may be more than one possibility; if so list them all.] (**1 Mark**)