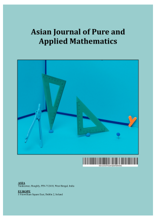

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Fuzzy Soft Prenormal Operators

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper introduces and systematically investigates a novel class of operators called fuzzy soft prenormal operators ($\mathcal{FSP}$) within the framework of fuzzy soft Hilbert spaces. Motivated by the need to extend classical operator theory to handle uncertainty and imprecision, we develop this class as a meaningful generalization of fuzzy soft normal operators. We establish several fundamental properties and characterizations, demonstrating that $\mathcal{FSP}$ operators preserve essential spectral features while offering enhanced flexibility in modeling operator behavior under fuzzy and parametric uncertainty. Key results include: the closure properties of $\mathcal{FSP}$ operators under addition and multiplication under specific commutation conditions; the invariance of the $\mathcal{FSP}$ property under translation by scalar multiples of the identity; topological closure in the strong operator topology; and the significant theorem that every fuzzy soft isometry satisfying the $\mathcal{FSP}$ condition is necessarily unitary. Furthermore, we explore the relationships between fuzzy soft prenormal operators and other established classes such as fuzzy soft normal, self-adjoint, and unitary operators. The theoretical contributions presented here not only enrich the landscape of fuzzy soft operator theory but also provide a robust foundation for potential applications in mathematical physics, engineering, and decision-making under uncertainty.

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1 Introduction

The study of operator theory has undergone extensive generalizations, particularly within the framework of fuzzy and soft set theories, as part of an ongoing effort to model uncertainty and imprecision in mathematical structures. The fusion of fuzzy theory and soft set theory has led to the emergence of the fuzzy soft Hilbert space, which provides a rich setting for extending classical operator-theoretic notions. Within this context, many classes of operators, such as normal, hyponormal, and quasi-hyponormal operators, have been revisited and extended to their fuzzy soft analogs, thereby enriching both theoretical and applied aspects of functional analysis.

The primary objective of this paper is to introduce and systematically analyze a new class of operators called fuzzy soft prenormal operators ($\mathcal{FSP}$). This class represents a meaningful extension of fuzzy soft normal operators, designed to capture operator behaviors that are intermediate between normality and hyponormality while maintaining essential spectral properties. We aim to establish fundamental characterizations, explore structural properties, and investigate relationships with existing operator classes within fuzzy soft Hilbert spaces. By doing so, we contribute to the expanding landscape of fuzzy soft operator theory and provide tools for modeling uncertainty in operator-theoretic frameworks.

The origin of these developments can be traced back to Zadeh's pioneering concept of fuzzy sets (Zadeh, 1965), introduced in 1965 to model vagueness through a membership function defined in a universal domain, and to Molodtsov's theory of soft sets (Molodtsov, 1999), proposed in 1999 as a flexible mathematical framework to handle parameterized uncertainties. Subsequent contributions expanded this foundation through constructs such as soft normed spaces Yazar et al. (2014), soft inner product spaces (Das and Samanta, 2013), and soft Hilbert spaces (Yazar et al., 2019). Later, researchers merged these two paradigms to define the fuzzy soft set (Maji et al., 2001), which inspired further studies on fuzzy soft points (Neog et al., 2012), fuzzy soft normed spaces (Beaula and Priyanga, 2015), fuzzy soft inner product spaces and fuzzy soft Hilbert spaces (Faried et al., 2020a). These culminated in the definition of the fuzzy soft linear operator and the fuzzy soft self-adjoint operator (Faried et al., 2020c,b), laying the groundwork for subsequent operator-theoretic explorations in the fuzzy soft setting.

Building on this foundation, Dawood and Jabur (2021) introduced the fuzzy soft normal operator, establishing essential properties and exploring its connections with related classes of operators such as self-adjoint and quasi-normal operators. This work provided a key stepping stone for the development of more complex fuzzy soft operator classes. Following this, (Assi and Kabban, 2024) formulated the concept of fuzzy soft quasi-normal operator $(\widetilde{T*}, \mathcal{N})$, giving several characterizations and analytical results that demonstrated its robustness within fuzzy soft Hilbert spaces. Extending this framework, (Mohsen, 2023) introduced the fuzzy soft $(\widetilde{k^*} - \mathcal{A})$-quasi-normal operator, presenting important theorems and relationships linking it to fuzzy soft Hermitian and fuzzy soft quasinormal operators, and providing a deeper understanding of the interplay between structure and generalization in fuzzy soft operator theory.

Further, Kadhim and Shubber (2025) investigated the fuzzy soft n-normal operator, analyzing its algebraic behavior and identifying conditions under which the addition and multiplication of such operators commute. Their results highlighted the structural flexibility of fuzzy soft n-normal operators and their potential applications in mathematical physics, medical science, and engineering. In a related advancement, Eidia and Mohsen (2022) introduced and examined the fuzzy soft $(n - \widetilde{N})$ quasi-normal operator, providing new theorems, operational conditions, and analytical properties that enrich the understanding of fuzzy soft quasi-normality. Together, these studies mark significant progress in generalizing the core concepts of normal and quasi-normal operators within the fuzzy soft environment.

Recent works by Mohsen and Mousa (2022) and Mohsen (2025) further strengthened this theoretical landscape through the development of fuzzy soft $\mathcal{M}$-hyponormal and fuzzy soft κ quasi-hyponormal operators, each contributing crucial analytic frameworks that link classical operator theory to its fuzzy soft counterparts. Their findings underscored the unifying potential of fuzzy soft Hilbert spaces in capturing a wide range of operator behaviors influenced by fuzziness and parametric uncertainty.

Motivated by these advances, the present work introduces a new class of operators called fuzzy soft prenormal operators. This class aims to generalize the concept of normality within the fuzzy soft framework by incorporating preconditions that balance between normal and hyponormal behaviors. The fuzzy soft prenormal operator preserves fundamental spectral and analytical characteristics while offering enhanced flexibility in modeling the interplay between fuzziness, softness, and operator behavior. Consequently, this study not only extends the frontier of fuzzy soft operator theory but also establishes a foundation for further exploration of spectral properties, equivalence relations, and stability phenomena in fuzzy soft Hilbert spaces.

1.1 Basic Concepts

This section presents the fundamental definitions and concepts that form the basis of fuzzy soft operator theory. We begin with the foundational notions of fuzzy sets and soft sets, progressing through fuzzy soft sets, fuzzy soft topological structures, and culminating in the definition of fuzzy soft Hilbert spaces and operators.

Definition 1.1. (Zadeh, 1965) Let $\widehat{S}$ be a fuzzy set over the universe set $\mathcal{X}$. It is characterized by a membership function

$$\mu_{\widehat{S}} : \mathcal{X} \longrightarrow \mathbb{T},$$

where $\mathbb{T} = [0, 1]$. The fuzzy set $\widehat{S}$ can be represented by an ordered pair

$$\widehat{S} = \{(x, \mu_{\widehat{S}}(x)) \mid x \in \mathcal{X},\ \mu_{\widehat{S}}(x) \in \mathbb{T}\},$$

or equivalently,

$$\widehat{S} = \left\{ \frac{\mu_{\widehat{S}}(x)}{x} : x \in \mathcal{X} \right\}.$$

Here, $\mu_{\widehat{S}}(x)$ is said to be the degree of membership of x in $\widehat{S}$. Moreover,

$$\mathfrak{T}_{\mathcal{X}} = \{\widehat{S} : \widehat{S} \text{ is a function from } \mathcal{X} \text{ into } \mathfrak{T}\}.$$

Example 1.1. *Consider a universe set $\mathcal{X} = \{x_1, x_2, x_3\}$ representing three different temperatures. A fuzzy set $\widehat{S}$ describing "comfortable temperature" might be defined as:*

$$\widehat{S} = \left\{ \frac{0.8}{x_1}, \frac{0.6}{x_2}, \frac{0.3}{x_3} \right\}$$

where x_1 has membership degree 0.8, x_2 has 0.6, and x_3 has 0.3 in the fuzzy set of comfortable temperatures.

Definition 1.2. (Molodtsov, 1999) Let $\mathcal{X}$ be a universe set, and $\mathcal{Q}$ be a set of parameters, $\mathcal{P}(\mathcal{X})$ the power set of $\mathcal{X}$ and $\mathcal{S} \subseteq \mathcal{Q}$. Suppose that $\mathcal{G}$ is a mapping given by

$$\mathcal{G} : \mathcal{S} \to \mathcal{P}(\mathcal{X}),$$

where

$$\mathcal{G}_{\mathcal{S}} = \{\mathcal{G}(q) \in \mathcal{P}(\mathcal{X}) : q \in \mathcal{S}\}.$$

The pair $(\mathcal{G}, \mathcal{S})$ or $\mathcal{G}_{\mathcal{S}}$ is called a **soft set** over $\mathcal{X}$ with respect to $\mathcal{S}$.

Example 1.2. *Let $\mathcal{X} = \{h_1, h_2, h_3\}$ be a set of houses and $\mathcal{S} = \{expensive, beautiful, wooden\}$ be a set of parameters. A soft set $(\mathcal{G}, \mathcal{S})$ describing house characteristics might be:*

$$\mathcal{G}(expensive) = \{h_1, h_3\}$$
$$\mathcal{G}(beautiful) = \{h_2, h_3\}$$
$$\mathcal{G}(wooden) = \{h_3\}$$

This represents that houses h_1 and h_3 are expensive, h_2 and h_3 are beautiful, and only h_3 is wooden.

Definition 1.3. (Maji et al., 2001) The soft set $(\mathcal{G}, S)$ is called a **fuzzy soft set** ($\mathcal{FS}$-set) over a universe set $\mathcal{X}$ whenever $\mathcal{G}$ is a mapping

$$\mathcal{G} : S \longrightarrow \mathbb{T}^{\mathcal{X}},$$

and

$$\{\mathcal{G}(q) \in \mathbb{T}^{\mathcal{X}} : q \in S\}.$$

The family of all $\mathcal{FS}$–sets is denoted by $\mathcal{FSS}(\tilde{\mathcal{X}})$.

Example 1.3. *Consider a medical diagnosis scenario where $\mathcal{X} = \{p_1, p_2, p_3\}$ represents patients and $S = \{fever, cough, fatigue\}$ represents symptoms. A fuzzy soft set could be:*

$$\mathcal{G}(fever) = \left\{\frac{0.9}{p_1}, \frac{0.4}{p_2}, \frac{0.7}{p_3}\right\}$$
$$\mathcal{G}(cough) = \left\{\frac{0.6}{p_1}, \frac{0.8}{p_2}, \frac{0.3}{p_3}\right\}$$
$$\mathcal{G}(fatigue) = \left\{\frac{0.7}{p_1}, \frac{0.5}{p_2}, \frac{0.9}{p_3}\right\}$$

This represents the degree to which each patient exhibits each symptom.

Definition 1.4. (Neog et al., 2012) The $\mathcal{FS}$–set $(\mathcal{G}, S) \in \mathcal{FSS}(\tilde{\mathcal{X}})$ is called a **fuzzy soft point** over $\mathcal{X}$, denoted by $((\tilde{x}_{\mu_{\mathcal{G}(q)}}, S))$ or $\tilde{x}_{\mu_{\mathcal{G}(q)}}$, if $q \in S$ and $x \in \mathcal{X}$, where

$$\mu_{\mathcal{G}(q)}(x) = \begin{cases} \lambda, & \text{if } x = x_0 \in \mathcal{X} \text{ and } q = q_0 \in S, \\ 0, & \text{if } x \in \mathcal{X} - \{x_0\} \text{ or } q \in S - \{q_0\}, \end{cases}$$

with $\lambda \in (0, 1]$.

Remark 1.1. (Neog et al., 2012) $\mathcal{C}(S)$ denotes the family of all $\mathcal{FS}$–complex numbers, and $\mathcal{R}(S)$ denotes the family of all $\mathcal{FS}$–real numbers.

Definition 1.5. (Beaula and Priyanga, 2015) Let $\tilde{\mathcal{X}}$ be a $\mathcal{FS}$–vector space. A mapping $\|\tilde{\cdot}\| : \tilde{\mathcal{X}} \to \mathcal{R}(S)$ is called a **fuzzy soft norm** on $\tilde{\mathcal{X}}$ (denoted $\mathcal{FSN}$) if it satisfies:

1. $\|\tilde{x}_{\mu_{\mathcal{G}(q)}}\| \geq \tilde{O}, \forall \tilde{x}_{\mu_{\mathcal{G}(q)}} \in \tilde{\mathcal{X}}$, and $\|\tilde{x}_{\mu_{\mathcal{G}(q)}}\| = \tilde{O} \iff \tilde{x}_{\mu_{\mathcal{G}(q)}} = \tilde{\theta}$.
2. $\|\tilde{r} \cdot \tilde{x}_{\mu_{\mathcal{G}(q)}}\| = |\tilde{r}| \, \|\tilde{x}_{\mu_{\mathcal{G}(q)}}\|$, $\forall \tilde{r} \in \mathcal{C}(S)$.
3. $\|\tilde{x}_{\mu_1 \mathcal{G}(q_1)} + \tilde{y}_{\mu_2 \mathcal{G}(q_2)}\| \leq \|\tilde{x}_{\mu_1 \mathcal{G}(q_1)}\| + \|\tilde{y}_{\mu_2 \mathcal{G}(q_2)}\|$, $\forall \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{y}_{\mu_2 \mathcal{G}(q_2)} \in \tilde{\mathcal{X}}$.

The $\mathcal{FS}$–vector space with fuzzy soft norm $\|\tilde{\cdot}\|$ is called a **fuzzy soft normed vector space** ($\mathcal{FSN}$–space) and is denoted by $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$.

Definition 1.6. (Faried et al., 2020a) Let $\tilde{\mathcal{X}}$ be a $\mathcal{FSV}$–space. A mapping

$$\langle \tilde{\cdot}, \tilde{\cdot} \rangle : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \to (\mathcal{C}(S) \text{ or } \mathcal{R}(S))$$

is called a **fuzzy soft inner product** on $\tilde{\mathcal{X}}$ (denoted $\mathcal{FST}$) if it satisfies:

1. $\langle \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{x}_{\mu_1\mathcal{G}(q_1)} \rangle \geq \tilde{O}$, and equals $\tilde{O}$ iff $\tilde{x}_{\mu_1\mathcal{G}(q_1)} = \tilde{\theta}$.
2. $\langle \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle = \overline{\langle \tilde{y}_{\mu_2\mathcal{G}(q_2)}, \tilde{x}_{\mu_1\mathcal{G}(q_1)} \rangle}$.
3. $\langle \tilde{\alpha}\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle = \tilde{\alpha}\langle \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle$, for all $\tilde{\alpha} \in \mathcal{C}(S)$.
4. $\langle \tilde{x}_{\mu_1\mathcal{G}(q_1)} + \tilde{y}_{\mu_2\mathcal{G}(q_2)}, \tilde{z}_{\mu_3\mathcal{G}(q_3)} \rangle = \langle \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{z}_{\mu_3\mathcal{G}(q_3)} \rangle + \langle \tilde{y}_{\mu_2\mathcal{G}(q_2)}, \tilde{z}_{\mu_3\mathcal{G}(q_3)} \rangle$.

The $\mathcal{FS}$–vector space $\tilde{\mathcal{X}}$ with $\mathcal{FST}$ is called a **fuzzy soft inner product space** ($\mathcal{FST}$–space), denoted by $(\tilde{\mathcal{X}}, \langle \tilde{\cdot}, \tilde{\cdot} \rangle)$.

Definition 1.7. (Khameneh et al., 2013) A sequence of $\mathcal{FS}$–vectors $\{\tilde{x}_{\mu_n\mathcal{G}(q_n)}\}_n$ in the $\mathcal{FSN}$–space $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$ is called **fuzzy soft convergent** and converges to $\tilde{x}_{\mu_0\mathcal{G}(q_0)}$ if

$$\lim_{n\to\infty} \|\tilde{x}_{\mu_n\mathcal{G}(q_n)} - \tilde{x}_{\mu_0\mathcal{G}(q_0)}\| = \tilde{O},$$

i.e., $\forall \tilde{\varepsilon} > \tilde{O}, \exists n_0 \in \mathbb{N}$ such that

$$\|\tilde{x}_{\mu_n\mathcal{G}(q_n)} - \tilde{x}_{\mu_0\mathcal{G}(q_0)}\| < \tilde{\varepsilon},\ \forall n \geq n_0.$$

It is denoted by

$$\lim_{n\to\infty} \tilde{x}_{\mu_n\mathcal{G}(q_n)} = \tilde{x}_{\mu_0\mathcal{G}(q_0)} \quad \text{or} \quad \tilde{x}_{\mu_n\mathcal{G}(q_n)} \to \tilde{x}_{\mu_0\mathcal{G}(q_0)} \text{ as } n \to \infty.$$

Definition 1.8. (Khameneh et al., 2013) A sequence of $\mathcal{FS}$–vectors $\{\tilde{x}_{\mu_n\mathcal{G}(q_n)}\}_n$ in the $\mathcal{FSN}$–space $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$ is called a **fuzzy soft Cauchy sequence** if for every $\tilde{\varepsilon} > \tilde{O}$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\tilde{x}_{\mu_n\mathcal{G}(q_n)} - \tilde{x}_{\mu_m\mathcal{G}(q_m)}\| < \tilde{\varepsilon}, \quad \forall n, m \geq n_0,\ n > m.$$

That is,

$$\|\tilde{x}_{\mu_n\mathcal{G}(q_n)} - \tilde{x}_{\mu_m\mathcal{G}(q_m)}\| \to \tilde{O} \quad \text{as } n, m \to \infty.$$

Definition 1.9. (Khameneh et al., 2013) The $\mathcal{FSN}$–space $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$ is called **fuzzy soft complete** (or $\mathcal{FS}$**–complete**) if every $\mathcal{FS}$–Cauchy sequence is a $\mathcal{FS}$–convergent sequence in it.

Definition 1.10. (Faried et al., 2020a) The $\mathcal{FS}$–space $(\tilde{\mathcal{X}}, \langle \tilde{\cdot}, \tilde{\cdot} \rangle)$ is called a **fuzzy soft Hilbert space** ($\mathcal{FSH}$–space) if it is $\mathcal{FS}$–complete in the induced $\mathcal{FSN}$ given by

$$\|\tilde{x}_{\mu_{\mathcal{G}(q)}}\| = \sqrt{\langle \tilde{x}_{\mu_{\mathcal{G}(q)}}, \tilde{x}_{\mu_{\mathcal{G}(q)}} \rangle}.$$

It is denoted by $(\tilde{\mathcal{H}}, \langle \tilde{\cdot}, \tilde{\cdot} \rangle)$.

Definition 1.11. (Faried et al., 2020c) Let $\tilde{\mathcal{H}}$ be a $\mathcal{FSH}$–space and $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a fuzzy soft operator. Then $\tilde{\mathcal{T}}$ is called a **fuzzy soft linear operator** ($\mathcal{FSL}$–operator) if:

1. $\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)} + \tilde{y}_{\mu_2\mathcal{G}(q_2)}) = \tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) + \tilde{\mathcal{T}}(\tilde{y}_{\mu_2\mathcal{G}(q_2)})$, for all $\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \in \tilde{\mathcal{H}}$.
2. $\tilde{\mathcal{T}}(\tilde{\beta}\tilde{x}_{\mu_1\mathcal{G}(q_1)}) = \tilde{\beta}\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)})$, for all $\tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}}$ and $\tilde{\beta} \in \mathcal{C}(S)$.

That is,

$$\tilde{\mathcal{T}}(\tilde{\alpha}\tilde{x}_{\mu_1\mathcal{G}(q_1)} + \tilde{\beta}\tilde{y}_{\mu_2\mathcal{G}(q_2)}) = \tilde{\alpha}\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) + \tilde{\beta}\tilde{\mathcal{T}}(\tilde{y}_{\mu_2\mathcal{G}(q_2)}),$$

for all $\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \in \tilde{\mathcal{H}}$ and fuzzy soft scalars $\tilde{\alpha}, \tilde{\beta}$.

Definition 1.12. (Faried et al., 2020c) Let $\tilde{\mathcal{H}}$ be a $\mathcal{FSH}$–space and $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a $\mathcal{FS}$–operator. Then $\tilde{\mathcal{T}}$ is called a **fuzzy soft bounded operator** ($\mathcal{FSB}$–operator) if there exists $\tilde{m} \in \mathcal{R}(S)$ such that

$$\|\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)})\| \leq \tilde{m}\|\tilde{x}_{\mu_1\mathcal{G}(q_1)}\|, \quad \forall \tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}}.$$

The collection of all $\mathcal{FS}$–linear and bounded operators is denoted by $\tilde{\mathcal{B}}(\tilde{\mathcal{H}})$.

Definition 1.13. (Faried et al., 2020c) Let $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$ be two $\mathcal{FSH}$–spaces. Then:

1. The **range** of the $\mathcal{FS}$–operator $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{K}}$ is denoted by $\mathrm{Ran}(\tilde{\mathcal{T}})$ and defined as

$$\mathrm{Ran}(\tilde{\mathcal{T}}) = \{\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) \in \tilde{\mathcal{K}} : \tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}}\}.$$

2. The **null space** (kernel) of $\tilde{\mathcal{T}}$ is denoted by $\mathrm{Ker}(\tilde{\mathcal{T}})$ and defined as

$$\mathrm{Ker}(\tilde{\mathcal{T}}) = \{\tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}} \ : \ \tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) = \tilde{0}\}.$$

Example 1.4. *(Faried et al., 2020c) The $\mathcal{FS}$–operator $\tilde{\mathcal{I}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ defined by*

$$\tilde{\mathcal{I}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) = \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \quad \forall \tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}},$$

*is called the **fuzzy soft identity operator**.*

Definition 1.14. (Faried et al., 2020c) Let $\tilde{\mathcal{H}}$ be a $\mathcal{FSH}$–space and $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a $\mathcal{FSB}$–operator. Then the **fuzzy soft adjoint operator $\tilde{\mathcal{T}}^*$** is defined by

$$\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle = \langle \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{\mathcal{T}}^*\tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle, \quad \forall \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \in \tilde{\mathcal{H}}.$$

Definition 1.15. (Faried et al., 2020b) The $\mathcal{FS}$–operator $\widetilde{\mathcal{T}}$ of the $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ is called a **fuzzy soft self-adjoint operator** ($\mathcal{FS}$–self-adjoint operator) if

$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}^*.$$

Definition 1.16. (Dawood and Jabur, 2021) Let $\widetilde{\mathcal{T}}$ be an $\mathcal{FS}$–operator on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$. Then $\widetilde{\mathcal{T}}$ is called a **fuzzy soft normal operator** ($\mathcal{FSN}$–operator) if

$$\widetilde{\mathcal{T}}\,\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\,\widetilde{\mathcal{T}}.$$

Definition 1.17. (Dawood and Jabur, 2021) A $\mathcal{FS}$–operator $\widetilde{\mathcal{U}}$ on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ is called a **fuzzy soft unitary operator** ($\mathcal{FSU}$–operator) if

$$\widetilde{\mathcal{U}}\widetilde{\mathcal{U}}^* = \widetilde{\mathcal{U}}^*\widetilde{\mathcal{U}} = \widetilde{\mathcal{I}}.$$

2 Main Results

This section presents the core contributions of our work. We begin by introducing the novel concept of fuzzy soft prenormal operators ($\mathcal{FSP}$) and subsequently establish their fundamental properties, characterizations, and relationships with existing operator classes. The results are systematically organized into definitions, theorems, propositions, and illustrative examples to provide a comprehensive understanding of this new operator class.

2.1 Fuzzy Soft Prenormal Operators: Definition and Basic Properties

Definition 2.1. Let $\widetilde{\mathcal{T}}$ be an $\mathcal{FS}$–operator on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$. Then $\widetilde{\mathcal{T}}$ is called a **fuzzy soft prenormal operator** ($\mathcal{FSP}$–operator) if

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2.$$

Remark 2.1. The condition $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$ represents a weaker form of normality compared to the standard fuzzy soft normal condition $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$. While every fuzzy soft normal operator is necessarily a fuzzy soft prenormal operator, the converse is not generally true. This makes the $\mathcal{FSP}$ class more inclusive while preserving many desirable spectral properties.

Example 2.1. *Consider a fuzzy soft Hilbert space $\widetilde{\mathcal{H}}$ and let $\widetilde{\mathcal{T}}$ be defined by the fuzzy soft matrix representation:*

$$\widetilde{\mathcal{T}} = \begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{b} \end{pmatrix}$$

where $\tilde{a}, \tilde{b} \in \mathcal{C}(S)$ with $|\tilde{a}|^4 = |\tilde{b}|^4$. Then:

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \begin{pmatrix} |\tilde{a}|^2 & \tilde{0} \\ \tilde{0} & |\tilde{b}|^2 \end{pmatrix}, \quad \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} = \begin{pmatrix} |\tilde{a}|^2 & \tilde{0} \\ \tilde{0} & |\tilde{b}|^2 \end{pmatrix}$$

Clearly, $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$, so $\widetilde{\mathcal{T}}$ is an $\mathcal{FSP}$–operator. Note that if $|\tilde{a}|^2 \neq |\tilde{b}|^2$, then $\widetilde{\mathcal{T}}$ is not normal but still prenormal.

Theorem 2.2 (Algebraic Closure Properties)**.** *Let $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{S}}$ be $\mathcal{FSP}$–operators on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ satisfying*

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}^* = \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}} \quad \text{and} \quad \widetilde{\mathcal{S}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}}.$$

Then both $\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}$ are $\mathcal{FSP}$–operators.

Proof. Since $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{S}}$ are $\mathcal{FSP}$–operators, we have

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2, \qquad (\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)^2 = (\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})^2. \tag{1}$$

Also, by assumption,

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}^* = \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}}, \qquad \widetilde{\mathcal{S}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}}. \tag{2}$$

(i) To show that $\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}$ is an $\mathcal{FSP}$–operator.

We first compute

$$(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* + \widetilde{\mathcal{T}}\widetilde{\mathcal{S}}^* + \widetilde{\mathcal{S}}\widetilde{\mathcal{T}}^* + \widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*.$$

Using (2), this becomes

$$(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}} + \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}} + \widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*. \tag{3}$$

Similarly,

$$(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}) = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}} + \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}} + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}. \tag{4}$$

Now, squaring both sides of (3) and (4), and applying (1) together with the commutation relations in (2), we obtain

$$\left[(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^*\right]^2 = \left[(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})\right]^2.$$

Hence, $\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}$ is an $\mathcal{FSP}$–operator.

(ii) To show that $\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}$ is an $\mathcal{FSP}$–operator. We have

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}(\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)\widetilde{\mathcal{T}}^*.$$

By (1),

$$(\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)^2 = (\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})^2,$$

and so

$$\left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*\right]^2 = \widetilde{\mathcal{T}}(\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)^2\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}(\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})^2\widetilde{\mathcal{T}}^*.$$

Moreover,

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}^*(\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})\widetilde{\mathcal{S}},$$

and hence

$$\left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})\right]^2 = \widetilde{\mathcal{S}}^*(\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2\widetilde{\mathcal{S}}.$$

Since $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$, we finally get

$$\left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*\right]^2 = \left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})\right]^2.$$

Therefore, $\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}$ is also an $\mathcal{FSP}$–operator. $\square$

Theorem 2.3 (Norm Characterization). *An $\mathcal{FS}$–operator $\widetilde{\mathcal{T}}$ on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ is an $\mathcal{FSN}$–operator if and only if*

$$\|\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\| \quad \textit{for every } \widetilde{x}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}.$$

Proof. ($\Rightarrow$) Suppose that $\widetilde{\mathcal{T}}$ is an $\mathcal{FSN}$–operator, that is, $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$. Then, for any $\widetilde{x}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}$, we have

$$\|\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\|^2 = \langle \widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle = \langle \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle.$$

Since $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$, it follows that

$$\langle \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle = \langle \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle = \|\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\|^2.$$

Hence,

$$\|\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\|.$$

($\Leftarrow$) Conversely, suppose that

$$\|\widetilde{\mathcal{T}}\widetilde{x}\| = \|\widetilde{\mathcal{T}}^*\widetilde{x}\| \quad \text{for every } \widetilde{x} \in \widetilde{\mathcal{H}}.$$

Then

$$\langle (\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})\widetilde{x}, \widetilde{x}\rangle = 0, \quad \forall\, \widetilde{x} \in \widetilde{\mathcal{H}}.$$

Since $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$ is self–adjoint, it must be the zero operator. Therefore,

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}},$$

and thus $\widetilde{\mathcal{T}}$ is an $\mathcal{FSN}$–operator. □

Theorem 2.4 (Translation Invariance). *Let $\widetilde{\mathcal{T}}$ be an $\mathcal{FSP}$–operator on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ and let $\widetilde{\alpha} \in \mathcal{C}(A)$. Then the operator $\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I}$ is also an $\mathcal{FSP}$–operator.*

Proof. Since $\widetilde{\mathcal{T}}$ is an $\mathcal{FSP}$–operator, we have

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2.$$

Consider $\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I}$:

$$\big((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*\big)^2 = \big((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})(\widetilde{\mathcal{T}}^* - \widetilde{\overline{\alpha}}\widetilde{I})\big)^2.$$

Expanding, we get

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* - \widetilde{\alpha}\widetilde{\mathcal{T}}^* - \widetilde{\overline{\alpha}}\widetilde{\mathcal{T}} + |\widetilde{\alpha}|^2\widetilde{I})^2.$$

Similarly,

$$\big((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})\big)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{\mathcal{T}}^* - \widetilde{\overline{\alpha}}\widetilde{\mathcal{T}} + |\widetilde{\alpha}|^2\widetilde{I})^2.$$

Since $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$ and scalar multiples commute with operators, it follows that

$$\big((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*\big)^2 = \big((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})\big)^2.$$

Hence, $\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I}$ is also an $\mathcal{FSP}$–operator. □

Proposition 2.1 (Topological Closure). *The class of $\mathcal{FSP}$–operators is closed in the strong operator topology.*

Proof. Let $\{\widetilde{\mathcal{T}}_n\}$ be a sequence of $\mathcal{FSP}$–operators on $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ that converges strongly to $\widetilde{\mathcal{T}} \in \widetilde{\mathcal{B}}(\widetilde{\mathcal{H}})$. That is,

$$\widetilde{\mathcal{T}}_n\widetilde{x} - \widetilde{\mathcal{T}}\widetilde{x} \to 0 \quad \text{as } n \to \infty \text{ for each } \widetilde{x} \in \widetilde{\mathcal{H}}.$$

Since each $\widetilde{\mathcal{T}}_n$ is $\mathcal{FSP}$, we have

$$(\widetilde{\mathcal{T}}_n\widetilde{\mathcal{T}}_n^*)^2 = (\widetilde{\mathcal{T}}_n^*\widetilde{\mathcal{T}}_n)^2.$$

By the continuity of operator multiplication in the strong operator topology,

$$(\widetilde{\mathcal{T}}_n\widetilde{\mathcal{T}}_n^*)^2 \to (\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 \quad \text{and} \quad (\widetilde{\mathcal{T}}_n^*\widetilde{\mathcal{T}}_n)^2 \to (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2 \quad \text{strongly as } n \to \infty.$$

Therefore, taking the limit as $n \to \infty$ in the equality

$$(\widetilde{\mathcal{T}}_n\widetilde{\mathcal{T}}_n^*)^2 = (\widetilde{\mathcal{T}}_n^*\widetilde{\mathcal{T}}_n)^2$$

gives

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2.$$

Hence, $\widetilde{\mathcal{T}}$ is an $\mathcal{FSP}$–operator. This proves that the class of $\mathcal{FSP}$–operators is closed in the strong operator topology. □

Theorem 2.5 (Isometry Characterization). *If $\widetilde{\mathcal{T}} \in (\mathcal{FSP})$ is a fuzzy soft isometry, then $\widetilde{\mathcal{T}}$ is a fuzzy soft unitary operator.*

Proof. Since $\widetilde{\mathcal{T}} \in (\mathcal{FSP})$, by definition of a fuzzy soft prenormal operator, there exists a fuzzy soft unitary operator $\widetilde{\mathcal{U}}$ on the fuzzy soft Hilbert space $\widetilde{\mathcal{H}}$ such that

$$(\widetilde{\mathcal{T}}^{*2})(\widetilde{\mathcal{T}}^2) = \widetilde{\mathcal{U}}(\widetilde{\mathcal{T}}^{*2}\widetilde{\mathcal{T}}^2)\widetilde{\mathcal{U}}^* \quad \text{and} \quad (\widetilde{\mathcal{T}}^2)(\widetilde{\mathcal{T}}^{*2}) = \widetilde{\mathcal{U}}(\widetilde{\mathcal{T}}^2\widetilde{\mathcal{T}}^{*2})\widetilde{\mathcal{U}}^*.$$

Since $\widetilde{\mathcal{T}}$ is a fuzzy soft isometry, we have

$$\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} = \widetilde{\mathcal{I}},$$

where $\widetilde{\mathcal{I}}$ denotes the fuzzy soft identity operator.
Then,

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}(\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{I}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*.$$

By the $\mathcal{FSP}$ property, it follows that

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{I}}.$$

Hence,

$$\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{I}}.$$

Therefore, $\widetilde{\mathcal{T}}$ is a fuzzy soft unitary operator. □

Remark 2.2. The relationships between various classes of fuzzy soft operators can be summarized as follows:

$$\text{Fuzzy Soft Unitary} \subset \text{Fuzzy Soft Normal} \subset \text{Fuzzy Soft Prenormal} \subset \tilde{\mathcal{B}}(\widetilde{\mathcal{H}})$$

Each inclusion is strict, and the $\mathcal{FSP}$ class properly contains the normal operators while maintaining many of their desirable properties.

3 Discussion and Potential Applications

3.1 Theoretical Implications

The introduction of fuzzy soft prenormal operators represents a significant advancement in fuzzy soft operator theory. Unlike the strict condition of normality which requires $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$, the prenormal condition $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$ offers greater flexibility while preserving essential spectral characteristics. This relaxation allows for the modeling of operators that exhibit "approximate normality" or operators whose normality properties emerge only at the level of their squares.

The closure properties established in Theorem 2.2 demonstrate that $\mathcal{FSP}$ operators form a robust algebraic structure under appropriate commutation conditions. This structural stability is crucial for building more complex operator systems from simpler components. Furthermore, the topological closure property shown in Proposition 2.1 ensures that limits of sequences of $\mathcal{FSP}$ operators remain within the class, which is essential for approximation theory and numerical methods.

The characterization of isometries within the $\mathcal{FSP}$ class (Theorem 2.5) reveals an interesting connection: every fuzzy soft isometry that satisfies the prenormal condition is necessarily unitary. This result bridges the gap between metric preservation properties and algebraic normality conditions, providing new insights into the structure of fuzzy soft Hilbert space operators.

3.2 Potential Applications

The theory of fuzzy soft prenormal operators opens several avenues for practical applications across various domains:

1. **Quantum Computing with Uncertainty**: In quantum information processing, operators often need to handle uncertain or imprecise states. Fuzzy soft prenormal operators could model quantum gates or measurements where uncertainty is inherent in the system parameters. The relaxed normality condition allows for more flexible quantum operations while maintaining essential unitarity-like properties.
2. **Signal Processing under Vagueness**: Traditional signal processing assumes precise numerical values, but real-world signals often contain vagueness and parameter uncertainty. $\mathcal{FSP}$ operators could be employed in fuzzy soft wavelet transforms or fuzzy soft Fourier analysis where the transform operators need to handle imprecise frequency or time parameters.
3. **Medical Diagnosis Systems**: As shown in Example 1.3, fuzzy soft sets effectively model medical symptoms with degrees of membership. $\mathcal{FSP}$ operators could represent diagnostic transformations that map symptom patterns to disease probabilities, handling the inherent uncertainty in medical observations while maintaining mathematical consistency.
4. **Decision Support Systems**: In multi-criteria decision making with fuzzy parameters, $\mathcal{FSP}$ operators could model the aggregation processes that combine multiple uncertain criteria. The algebraic properties ensure that combinations of such aggregation operators behave predictably.
5. **Control Systems with Imprecise Parameters**: For control systems operating under uncertain conditions, $\mathcal{FSP}$ operators could represent state transition matrices that incorporate fuzzy parameters, allowing for more robust control design that accounts for measurement imprecision.
6. **Image Processing with Fuzzy Features**: In computer vision and image analysis, features often have fuzzy boundaries. $\mathcal{FSP}$ operators could implement image transformations that preserve essential structures while handling the fuzziness in feature detection and representation.

3.3 Future Research Directions

Several promising research directions emerge from this work:

1. **Spectral Theory of $\mathcal{FSP}$ Operators**: Developing a comprehensive spectral theory for fuzzy soft prenormal operators, including spectral mapping theorems and functional calculus.
2. **Perturbation Theory**: Investigating how $\mathcal{FSP}$ properties behave under small perturbations, which is crucial for numerical stability and robustness analysis.
3. **Operator Algebras**: Studying the C^*-algebra generated by $\mathcal{FSP}$ operators and its representation theory.
4. **Generalizations**: Extending the concept to n-prenormal operators defined by $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^n = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^n$ for $n > 2$.
5. **Computational Methods**: Developing numerical algorithms for computing with $\mathcal{FSP}$ operators and their applications in practical problems.

4 Conclusions

From the foregoing results, it is evident that the class of fuzzy soft prenormal operators ($\mathcal{FSP}$) forms a stable and meaningful extension of the fuzzy soft normal operators. We have established several key properties:

1. The $\mathcal{FSP}$ class is algebraically closed under addition and multiplication when appropriate commutation conditions are satisfied (Theorem 2.2).
2. The property of being $\mathcal{FSP}$ is invariant under translation by scalar multiples of the identity (Theorem 2.4).
3. The class is topologically closed in the strong operator topology (Proposition 2.1).
4. Every fuzzy soft isometry satisfying the $\mathcal{FSP}$ condition is necessarily a fuzzy soft unitary operator (Theorem 2.5).

These results establish that the $\mathcal{FSP}$ framework preserves the essential spectral and structural properties associated with normality in the fuzzy soft Hilbert space setting while offering greater flexibility for modeling operators under uncertainty. The theoretical developments presented here not only enrich the landscape of fuzzy soft operator theory but also provide a robust foundation for potential applications across various scientific and engineering domains where uncertainty and imprecision are inherent.

Disclaimer (Artificial Intelligence)

The authors of this manuscript hereby declare that no generative Artificial Intelligence (AI) technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Authors have declared that no competing interests exist.

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