



# The Orbit Structure of the Product of Four Alternating Groups Acting on the Cartesian Product of Four Sets of Ordered Tuples

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*Authors' contributions*

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

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## Abstract

This paper explores the orbit structure of the direct product of four alternating groups acting on the Cartesian product of four sets of ordered  $\gamma$ -tuples. The associated orbitals and suborbit lengths are determined using combinatorial formulae. Contrary to previous studies involving ordered pairs or fewer group factors, this setting introduces significantly higher combinatorial configurations arising from higher-dimensional tuple structures. This brings rise to a general orbit structure pattern that do not arise in pair-based and fewer group factors cases. The number of orbitals of  $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$  acting on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$ , ( $\forall n - \gamma \geq 2$ ) is 16 and explicit suborbit lengths formulae are obtained using combinatorial methods. The results extend existing research on ranks and subdegrees while revealing new structural phenomena unique to actions on ordered  $\gamma$ -tuples of direct product of four alternating groups.

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## 1. Preliminaries

### 1.1 Introduction

The Cartesian product of sets of ordered  $\gamma$ -tuples  $U_1^{[\gamma]}, U_2^{[\gamma]}, U_3^{[\gamma]}$  and  $U_4^{[\gamma]}$  is defined as the set of all  $\gamma$ -tuples  $(u_1, u_2, u_3, u_4)$  such that  $u_1 \in U_1^{[\gamma]}, u_2 \in U_2^{[\gamma]}, u_3 \in U_3^{[\gamma]}$  and  $u_4 \in U_4^{[\gamma]}$ . The ordered sets are generated using the Groups, Algorithms and Programming software (GAP).

Higman (1964) introduced the rank of groups on finite permutation groups with a rank of 3. In 1970, Higman (1970) proved that the action of symmetric group  $S_n$  on 2-element subsets from the set  $P = \{1, 2, \dots, n\}$  is of rank 3 and the subdegrees are:  $1, 2(n-1)$  and  $\binom{n-1}{2}$ .

Nyaga *et al.*, (2011) proved that  $S_n$  acts transitively on  $P^{(r)}$  and the subdegrees are;  $1, \binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{n-r}{r-3}, \dots, \binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r}$ .

Nyaga (2018) proved that the action of direct product of  $A_n$ , is transitive on the Cartesian product of 3 sets. The rank and subdegrees associated with this action for  $n \geq 4$  is 8; and  $1, (n-1), (n-1)^2, (n-1)^3$  respectively.

Mutua *et al.*, (2018) showed that the action of direct product of  $S_n \times A_n$  on  $P \times Q$  to be both transitive and imprimitive for all  $n \geq 3$ . The associated rank for this action is 6 when  $n = 3$ , but is 4 for all  $n \geq 3$ . The subdegrees are:  $1, (n-1), (n-1), (n-1)^2$ .

Maraka *et al.*, (2025) established that the action of the direct product of 3 alternating groups acting on the Cartesian product of 3 sets of ordered tuples has  $2^3$  orbits and the suborbit lengths are:  $1, 3\left(\left(\frac{n!}{(n-\tau)!}\right) - 1\right), 3\left(\left(\frac{n!}{(n-\tau)!}\right) - 1\right)^2$  and  $\left(\left(\frac{n!}{(n-\tau)!}\right) - 1\right)^3$ .

Maraka (2025) determined the ranks and subdegrees of the action of the direct product of 4 alternating groups on the Cartesian product of 4 sets of ordered pairs, where the rank was determined to be 16 and subdegrees were  $1, 4\left(\left(\frac{n!}{(n-2)!}\right) - 1\right), 6\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^2, 4\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^3$  and  $\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^4$ . This research extends this line of study to the action on ordered  $\gamma$ -tuples. The resulting orbital structures are not direct analogues of the ordered pair case, but instead exhibit general combinatorial patterns arising from higher-dimensional tuple arrangements.

This study forms basis for future research on the suborbital graphs of this action and their theoretical properties.

### 1.2 Definitions and Theorems

#### Definition 1.2.1 Group action (Njagi, 2016)

Given a group  $G$  and a non-empty set  $U$ , the action of  $G$  to the left of  $U$  matches a unique element  $gu \in U$  if  $\forall g \in G$  such that for all  $u \in U$  and  $g_1, g_2 \in G$ :

- (i)  $(g_1g_2)u = g_1(g_2)u$ .
- (ii)  $\varrho.u = u$ , given that  $\varrho$  is the identity in  $G$ .

When  $G$  acts from the right side of  $U$ , its action can similarly be denoted as such.

**Definition 1.2.2 Transitive group (Kinyanjui, et al., 2013)**

A group  $G$  is termed to act transitively on a set  $U$  provided for all  $u, s \in U, \exists g \in G: g(u) = s$ , that is, the action gives only a one orbit.

**Definition 1.2.3 Stabilizer of an Element (Rose, 1978)**

Let  $u \in U$  and a group  $G$  act on  $U$ . The stabilizer of  $u$  in  $G$  is given by  $Stab_G(u) = \{g \in G: gu = u\}$ .

**Definition 1.2.4 Fixed point (Njagi, 2016)**

Given a non-empty set  $U$  and group  $G$  acting on  $U$  with  $g \in G$ . The set of elements  $u$  fixed by  $g \in G$  is referred to as fixed point set of  $h$  given by  $Fix(g) = \{u \in U: h(u) = u\}$ .

**Theorem 1.2.5 Orbit – Stabilizer Theorem (Rose, 1978)**

Given  $G$  acts on a set  $U, |Orb_G(u)| = |G: Stab_G(u)|$ .

**Definition 1.2.6 Direct product action (Cameron et al., 2008)**

Given  $(G_1, U_1)$  and  $(G_2, U_2)$  as permutation groups. The direct product  $G_1 \times G_2$  acts on the separate union  $U_1 \cup U_2$  by the law  $u(g_1, g_2) = \begin{cases} ug_1, & u \in U_1 \\ ug_2, & u \in U_2 \end{cases}$  and on Cartesian product  $U_1 \times U_2$  by the law  $(u_1, u_2)(g_1, g_2) = (u_1g_1, u_2g_2)$ .

**Theorem 1.2.7 (Maraka et al., 2024)**

The action of the product of finite alternating groups,  $A_{n_1} \times A_{n_2} \times \dots \times A_{n_{m-1}} \times A_{n_m}$ , acts transitively on the Cartesian product of finite sets of ordered  $\gamma$ -tuples,  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times \dots \times U_{m-1}^{[\gamma]} \times U_m^{[\gamma]}$  if and only if  $n - \gamma \geq 2$ .

**Definition 1.2.8 Orbit (Njagi, 2016)**

For  $u \in U$ , the action of a group  $G$  on  $U$ , partitions  $U$  into separate equivalence classes known as orbits. Hence,  $Orb_G(u) = \{gu: g \in G\}$ .

**Definition 1.2.9 Rank and Subdegrees (Nyaga et al., 2011)**

Given the action of  $G$  on a set  $U$  is transitive and  $u \in U$ . The orbits of  $G_u$  on  $U$  are referred to as the suborbits. The number of orbitals (rank) of  $G$  on  $U$  is the number of those suborbits and their sizes are called suborbit length (subdegrees) of  $G$  on  $U$ .

**Theorem 1.2.10 (Armstrong, 2013)**

The  $G_1 \times G_2 \times G_3$ -orbit containing  $(u_1, u_2, u_3) \in U_1 \times U_2 \times U_3$  is given by  $Orb_{G_1}(u_1) \times Orb_{G_2}(u_2) \times Orb_{G_3}(u_3)$  and the stabilizer of  $(u_1, u_2, u_3)$  is given by  $Stab_{G_1}(u_1) \times Stab_{G_2}(u_2) \times Stab_{G_3}(u_3)$ .

**2. Main Results**

**Theorem 2.1: The number of orbitals (Rank)**

If  $n - \gamma \geq 2$ , then the rank of  $G = A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$ , on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$  is 16.

**Proof:**

Let  $G$  act on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$ . Let  $[1,2,3, \dots, \gamma] \in U_1^{[\gamma]}$ ,  $[n+1, n+2, \dots, n+\gamma] \in U_2^{[\gamma]}$ ,  $[2n+1, 2n+2, \dots, 2n+\gamma] \in U_3^{[\gamma]}$  and  $[3n+1, 3n+2, \dots, 3n+\gamma] \in U_4^{[\gamma]}$ .

Let  $U = \{[1,2,3, \dots, \gamma], [n+1, n+2, \dots, n+\gamma], [2n+1, 2n+2, \dots, 2n+\gamma], [3n+1, 3n+2, \dots, 3n+\gamma]\}$ . Then,  $G_U$  has orbits with exactly 4,3, 2, 1 or no ordered  $\gamma$ -tuple elements from  $U$  which are given as follows in Table 1.

**Table 1. The rank of acting on  $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$  on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$**

Suborbit	Combinatorial formula	Number of suborbits
Orbit containing four ordered $\gamma$ -tuple from $U$	$4C_4$	1
Orbit containing three ordered $\gamma$ -tuple from $U$	$4C_3$	4
Orbit containing two ordered $\gamma$ -tuple from $U$	$4C_2$	6
Orbits containing one ordered $\gamma$ -tuple from $U$	$4C_1$	4
Orbit containing no ordered $\gamma$ -tuples from $U$	$4C_0$	1

The rank of  $G$  on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$  is;

$$R(G) = 4C_4 + 4C_3 + 4C_2 + 4C_1 + 4C_0 = 1 + 4 + 6 + 4 + 1.$$

$$R(G) = 16 = 2^4.$$

Therefore, there are  $2^4$  orbits.

**Theorem 2.2: The Suborbit lengths (Subdegrees)**

The subdegrees of  $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$ , on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$  are;  $1, \left(\frac{n!}{(n-\gamma)!} - 1\right), \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^3, \left(\frac{n!}{(n-\gamma)!} - 1\right)^4$ .

**Proof:**

Let  $G = A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$ , on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$ . Then,  $U_1^{[\gamma]} = \{[1,2,3, \dots, \gamma], \dots, [n, n-1, \dots, n-\gamma+1]\}$ ,  $U_2^{[\gamma]} = \{[n+1, n+2, \dots, n+\gamma], \dots, [2n, 2n-1, \dots, 2n-\gamma+1]\}$ ,  $U_3^{[\gamma]} = \{[2n+1, 2n+2, \dots, 2n+\gamma], \dots, [3n, 3n-1, \dots, 3n-\gamma+1]\}$  and  $U_4^{[\gamma]} = \{[3n+1, 3n+2, \dots, 3n+\gamma], \dots, [4n, 4n-1, \dots, 4n-\gamma+1]\}$ .

Let  $U_1 = \{[1,2,3, \dots, \gamma]\}$ , so  $Orb_{G_{U_1}} = Orb_{G_{\{[1,2,3, \dots, \gamma]\}}}$ ,  $U_2 = \{[n+1, n+2, \dots, n+\gamma]\}$ , so  $Orb_{G_{U_2}} = Orb_{G_{\{[n+1, n+2, \dots, n+\gamma]\}}}$ ,  $U_3 = \{[2n+1, 2n+2, \dots, 2n+\gamma]\}$ , so  $Orb_{G_{U_3}} = Orb_{G_{\{[2n+1, 2n+2, \dots, 2n+\gamma]\}}}$  and  $U_4 = \{[3n+1, 3n+2, \dots, 3n+\gamma]\}$ , so  $Orb_{G_{U_4}} = Orb_{G_{\{[3n+1, 3n+2, \dots, 3n+\gamma]\}}}$ .

Let  $U = \{U_1, U_2, U_3, U_4\}$ . By Theorem 1.1.10 we have;  $Orb_{G_U} = Orb_{G_{U_1}} \times Orb_{G_{U_2}} \times Orb_{G_{U_3}} \times Orb_{G_{U_4}}$ . Let;  $U_1^{[\gamma]'} = \{[1,2,3, \dots, \gamma]\} \setminus U_1^{[\gamma]} = \{[1,2,3, \dots, \gamma+1], \dots, [n, n-1, \dots, n-\gamma+1]\}$ ,  $U_2^{[\gamma]'} = \{[n+1, n+2, \dots, n+\gamma]\} \setminus U_2^{[\gamma]} = \{[n+1, n+2, \dots, n+\gamma+1], \dots, [2n, 2n-1, \dots, 2n-\gamma+1]\}$ ,  $U_3^{[\gamma]'} = \{[2n+1, 2n+2, \dots, 2n+\gamma]\} \setminus U_3^{[\gamma]} = \{[2n+1, 2n+2, \dots, 2n+\gamma+1], \dots, [3n, 3n-1, \dots, 3n-\gamma+1]\}$

$$\text{an}U_4^{[\gamma]'} = \{[3n + 1, 3n + 2, \dots, 3n + \gamma]\} \setminus U_3^{[\gamma]} = \{[3n + 1, 3n + 2, \dots, 3n + \gamma + 1], \dots, [4n, 4n - 1, \dots, 4n - \gamma + 1]\}.$$

$$|U_1^{[\gamma]}| = |U_2^{[\gamma]}| = |U_3^{[\gamma]}| = |U_4^{[\gamma]}| = \frac{n!}{(n - \gamma)!}.$$

$$\text{Thus, } |U_1^{[\gamma]'}| = |U_2^{[\gamma]'}| = |U_3^{[\gamma]'}| = |U_4^{[\gamma]'}| = \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right) \text{ and } |U_1| = |U_2| = |U_3| = |U_4| = 1.$$

The 16 orbits of  $Orb_{G_U}$  are:

**(a) The suborbit with exactly 4 ordered  $\gamma$  –tuples of elements from  $U$ .**

$$\Delta_0 = Orb_{G_U}\{U_1, U_2, U_3, U_4\}$$

$$|\Delta_0| = |U_1| \times |U_2| \times |U_3| \times |U_4|$$

Thus,  $|\Delta_0| = 1$ .

**(b) Suborbits with exactly 3 ordered  $\gamma$  –tuples of elements from  $U$ .**

$$\Delta_1 = Orb_{G_U}\{U_1', U_2, U_3, U_4\}: U_1' \in U_1^{[\gamma]'}$$

$$|\Delta_1| = |U_1^{[\gamma]'}| \times |U_2| \times |U_3| \times |U_4|$$

$$= \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right) \times 1 \times 1 \times 1$$

$$\text{Thus, } |\Delta_1| = \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right).$$

$$\Delta_2 = Orb_{G_U}\{U_1, U_2', U_3, U_4\}: U_2' \in U_2^{[\gamma]'}$$

$$|\Delta_2| = |U_1| \times |U_2^{[\gamma]'}| \times |U_3| \times |U_4|$$

$$= 1 \times \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right) \times 1 \times 1$$

$$\text{Thus, } |\Delta_2| = \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right).$$

$$\Delta_3 = Orb_{G_U}\{U_1, U_2, U_3', U_4\}: U_3' \in U_3^{[\gamma]'}$$

$$|\Delta_3| = |U_1| \times |U_2| \times |U_3^{[\gamma]'}| \times |U_4|$$

$$= 1 \times 1 \times \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right) \times 1$$

$$\text{Thus, } |\Delta_3| = \left( \left( \frac{n!}{(n - \gamma)!} \right) - 1 \right).$$

$$\Delta_4 = Orb_{G_U}\{U_1, U_2, U_3, U_4'\}: U_4' \in U_4^{[\gamma]'}$$

$$\begin{aligned} |\Delta_4| &= |U_1| \times |U_2| \times |U_3| \times |U_4^{[\gamma]'}| \\ &= 1 \times 1 \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \end{aligned}$$

$$\text{Thus, } |\Delta_4| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right).$$

**(c) Suborbits with exactly 2 ordered  $\gamma$  –tuples of elements from  $U$ .**

$$\Delta_5 = Orb_{G_U} \{U_1', U_2', U_3, U_4\}; U_1' \in U_1^{[\gamma]'}, U_2' \in U_2^{[\gamma]'}$$

$$\begin{aligned} |\Delta_5| &= |U_1^{[\gamma]'}| \times |U_2^{[\gamma]'}| \times |U_3| \times |U_4| \\ &= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \times 1 \end{aligned}$$

$$\text{Thus, } |\Delta_5| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2.$$

$$\Delta_6 = Orb_{G_U} \{U_1', U_2, U_3', U_4\}; U_1' \in U_1^{[\gamma]'}, U_3' \in U_3^{[\gamma]'}$$

$$\begin{aligned} |\Delta_6| &= |U_1^{[\gamma]'}| \times |U_2| \times |U_3^{[\gamma]'}| \times |U_4| \\ &= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \end{aligned}$$

$$\text{Thus, } |\Delta_6| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2.$$

$$\Delta_7 = Orb_{G_U} \{U_1', U_2, U_4, U_4'\}; U_1' \in U_1^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$$

$$\begin{aligned} |\Delta_7| &= |U_1^{[\gamma]'}| \times |U_2| \times |U_3| \times |U_4^{[\gamma]'}| \\ &= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \end{aligned}$$

$$\text{Thus, } |\Delta_7| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2.$$

$$\Delta_8 = Orb_{G_U} \{U_1, U_2', U_3', U_4\}; U_2' \in U_3^{[\gamma]'}, U_3' \in U_3^{[\gamma]'}$$

$$\begin{aligned} |\Delta_8| &= |U_1| \times |U_2^{[\gamma]'}| \times |U_3^{[\gamma]'}| \times |U_4| \\ &= 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \end{aligned}$$

Thus,  $|\Delta_8| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2$ .

$\Delta_9 = Orb_{G_U} \{U_1, U_2', U_3, U_4'\} : \in U_2' \in U_2^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$

$$|\Delta_9| = |U_1| \times |U_2^{[\gamma]'}| \times |U_3| \times |U_4^{[\gamma]'}|$$

$$= 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$$

Thus,  $|\Delta_9| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2$ .

$\Delta_{10} = Orb_{G_U} \{U_1, U_2, U_3', U_4'\} : \in U_3' \in U_3^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$

$$|\Delta_{10}| = |U_1| \times |U_2| \times |U_3^{[\gamma]'}| \times |U_4^{[\gamma]'}|$$

$$= 1 \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$$

Thus,  $|\Delta_{10}| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2$ .

**(d) Suborbits with exactly exactly 1 ordered  $\gamma$  –tuple of element from  $U$ .**

$\Delta_{11} = Orb_{G_P} \{U_1, U_2', U_3', U_4'\} : \in U_2' \in U_2^{[\gamma]'}, U_3' \in U_3^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$

$$|\Delta_{11}| = |U_1| \times |U_2^{[\gamma]'}| \times |U_3^{[\gamma]'}| \times |U_4^{[\gamma]'}|$$

$$= 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$$

Thus,  $|\Delta_{11}| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^3$ .

$\Delta_{12} = Orb_{G_U} \{U_1', U_2, U_3', U_4'\} : \in U_1' \in U_1^{[\gamma]'}, U_3' \in U_3^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$

$$|\Delta_{12}| = |U_2^{[\gamma]'}| \times |U_2| \times |U_3^{[\gamma]'}| \times |U_4^{[\gamma]'}|$$

$$= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$$

Thus,  $|\Delta_{12}| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^3$ .

$\Delta_{13} = Orb_{G_U} \{U_1', U_2', U_3, U_4'\} : \in U_1' \in U_1^{[\gamma]'}, U_2' \in U_2^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$

$$|\Delta_{13}| = |U_1^{[\gamma]'}| \times |U_2^{[\gamma]'}| \times |U_3| \times |U_4^{[\gamma]'}|$$

$$= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1 \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$$

Thus,  $|\Delta_{13}| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^3$ .

$$\Delta_{14} = Orb_{G_U}\{U_1', U_2', U_3', U_4\} : \in U_1' \in U_1^{[\gamma]'}, U_2' \in U_2^{[\gamma]'}, U_3' \in U_3^{[\gamma]'}$$

$$|\Delta_{14}| = |U_1^{[\gamma]'}| \times |U_2^{[\gamma]'}| \times |U_3^{[\gamma]'}| \times |U_4|$$

$$= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times 1$$

Thus,  $|\Delta_{14}| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^3$ .

**(e) Suborbits with no ordered  $\gamma$  –tuple of elements from  $U$ .**

$$\Delta_{15} = Orb_{G_U}\{U_1', U_2', U_3', U_4'\} : \in U_1' \in U_1^{[\gamma]'}, U_2' \in U_2^{[\gamma]'}, U_3' \in U_3^{[\gamma]'}, U_4' \in U_4^{[\gamma]'}$$

$$|\Delta_{15}| = |U_1^{[\gamma]'}| \times |U_2^{[\gamma]'}| \times |U_3^{[\gamma]'}| \times |U_4^{[\gamma]'}|$$

$$= \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) \times \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$$

Thus,  $|\Delta_{15}| = \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^4$ .

**Table 2. Subdegrees of  $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$  Acting on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$**

Length of suborbit	1	$\left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)$	$\left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2$	$\left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^3$	$\left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^4$
Number of suborbits	1	4	6	4	1

**Theorem 2.3 (Summation of subdegrees)**

The sum of the number of elements in all the 16  $U$  –orbits equals the cardinality of  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$ , that is;  $1 + 4 \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right) + 6 \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^2 + 4 \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^3 + \left( \left( \frac{n!}{(n-\gamma)!} \right) - 1 \right)^4 = |U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}|$ .

**Proof:**

Let  $a = \left( \frac{n!}{(n-\gamma)!} \right)$ . So, we have;

$$\begin{aligned}
 & 1 + 4(a - 1) + 6\{(a - 1)^2\} + 4\{(a - 1)^3\} + (a - 1)^4 \\
 & = 1 + 4a - 4 + 6a^2 - 12a + 6 + 4a^3 - 12a^2 + 12a - 4 + a^4 - 4a^3 + 6a^2 - 4a + 1 = a^4 \\
 \text{But, } a & = \left(\frac{n!}{(n-\gamma)!}\right), \text{ so } a^4 = \left(\frac{n!}{(n-\gamma)!}\right)^4 = |U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}|.
 \end{aligned}$$

**Example 2.4**

The rank of  $A_4 \times A_4 \times A_4 \times A_4$  acting on  $U_1^{[2]} \times U_2^{[2]} \times U_3^{[2]} \times U_4^{[2]}$  is 16 and the subdegrees are; 1, 11,11,11,11,121,121,121,121,121,121,1 331,1 331,1 331,1 331 and 14 641.

**Proof:**

From Theorem 1.1,  $n - \gamma \geq 2$ , that,  $4 > 2$  and the rank,  $R(G)$ , is given by;

$$R(G) = 4C_4 + 4C_3 + 4C_2 + 4C_1 + 4C_0 = 1 + 4 + 6 + 4 + 1.$$

$$R(G) = 16.$$

Let  $G = A_4 \times A_4 \times A_4 \times A_4$  act on  $U_1^{[2]} \times U_2^{[2]} \times U_3^{[2]} \times U_4^{[2]}$ .

Then,  $U_1^{[2]} = \text{gap} > \text{Arrangements}([1,2,3,4], 2)$ ;

$$\{[1, 2], [1, 3], [1, 4], [2, 1], [2, 3], [2, 4], [3, 1], [3, 2], [3, 4], [4, 1], [4, 2], [4, 3]\},$$

$$U_2^{[2]} = \text{gap} > \text{Arrangements}([5,6,7,8], 2)$$

$$\{[5, 6], [5, 7], [5, 8], [6, 5], [6, 7], [6, 8], [7, 5], [7, 6], [7, 8], [8, 5], [8, 6], [8, 7]\},$$

$$U_3^{[2]} = \text{gap} > \text{Arrangements}([9,10,11,12], 2)$$

$$\{[9, 10], [9, 11], [9, 12], [10, 9], [10, 11], [10, 12], [11, 9], [11, 10], [11, 12], [12, 9], [12, 10], [12, 11]\}$$

and;

$$U_4^{[2]} = \text{gap} > \text{Arrangements}([13,14,15,16], 2)$$

$$\{[13, 14], [13, 15], [13, 16], [14, 13], [14, 15], [14, 16], [15, 13], [15, 14], [15, 16], [16, 13], [16, 14], [16, 15]\}.$$

Let  $U_1 = [1, 2]$ ,  $U_2 = [5,6]$ ,  $U_3 = [9, 10]$  and  $U_4 = [13,14]$ , . So,  $U = \{U_1, U_2, U_3, U_4\}$ .

$$U_1^{[2]'} = U_1 \setminus U_1^{[2]} = \{[1, 3], [1, 4], [2, 1], [2, 3], [2, 4], [3, 1], [3, 2], [3, 4], [4, 1], [4, 2], [4, 3]\},$$

$$U_2^{[2]'} = U_2 \setminus U_2^{[2]} = \{[5, 7], [5, 8], [6, 5], [6, 7], [6, 8], [7, 5], [7, 6], [7, 8], [8, 5], [8, 6], [8, 7]\},$$

$$U_3 \setminus U_3^{[2]} = \{[9, 11], [9, 12], [10, 9], [10, 11], [10, 12], [11, 9], [11, 10], [11, 12], [12, 9], [12, 10], [12, 11]\}$$

and;

$$U_4 \setminus U_4^{[2]}$$

$$= \{[13, 15], [13, 16], [14, 13], [14, 15], [14, 16], [15, 13], [15, 14], [15, 16], [16, 13], [16, 14], [16, 15]\}.$$

$$|U_1^{[2]}| = |U_2^{[2]}| = |U_3^{[2]}| = |U_4^{[2]}| = \frac{4!}{(4-2)!} = 12$$

$$|U_1^{[2]'}| = |U_2^{[2]'}| = |U_3^{[2]'}| = |U_4^{[2]'}| = \left( \left( \frac{4!}{(4-2)!} \right) - 1 \right) = 11$$

By Theorem 1.1.10, we have;  $Orb_{G_U} = Orb_{G_{U_1}} \times Orb_{G_{U_2}} \times Orb_{G_{U_3}} \times Orb_{G_{U_4}}$ .  
Therefore, the 16 orbits of  $Orb_{G_U}$  are;

**(a) The suborbit with exactly 4 ordered  $\gamma$  –tuples of elements from  $U$ .**

$$\Delta_0 = Orb_{G_U}\{U_1, U_2, U_3, U_4\}$$

$$|\Delta_0| = |U_1| \times |U_2| \times |U_3| \times |U_4|$$

Thus,  $|\Delta_0| = 1$ .

**(b) Suborbits with exactly 3 ordered  $\gamma$  –tuples of elements from  $U$ .**

$$\Delta_1 = Orb_{G_U}\{U_1', U_2, U_3, U_4\}; P_1' \in U_1^{[2]'}$$

$$|\Delta_1| = |U_1^{[2]'}| \times |U_2| \times |U_3| \times |U_4|$$

$$= 11 \times 1 \times 1 \times 1$$

Thus,  $|\Delta_1| = 11$ .

$$\Delta_2 = Orb_{G_U}\{U_1, U_2', U_3, U_4\}; U_2' \in U_2^{[2]'}$$

$$|\Delta_2| = |U_1| \times |U_2^{[2]'}| \times |U_3| \times |U_4|$$

$$= 1 \times 11 \times 1 \times 1$$

Thus,  $|\Delta_2| = 11$ .

$$\Delta_3 = Orb_{G_U}\{U_1, U_2, U_3', U_4\}; U_3' \in U_3^{[2]'}$$

$$|\Delta_3| = |U_1| \times |U_2| \times |U_3^{[2]'}| \times |U_4|$$

$$= 1 \times 1 \times 11 \times 1$$

Thus,  $|\Delta_3| = 11$ .

$$\Delta_4 = Orb_{G_U}\{U_1, U_2, U_3, U_4'\}; U_4' \in U_4^{[2]'}$$

$$|\Delta_4| = |U_1| \times |U_2| \times |U_3| \times |U_4^{[2]'}|$$

$$= 1 \times 1 \times 1 \times 11$$

Thus,  $|\Delta_4| = 11$ .

**(c) Suborbits with exactly 2 ordered  $\gamma$  –tuples of elements from  $U$ .**

$$\Delta_5 = Orb_{G_U}\{U_1', U_2', U_3, U_4\}: U_1' \in U_1^{[2]'}, U_2' \in U_2^{[2]'}$$

$$|\Delta_5| = |U_1^{[2]'}| \times |U_2^{[2]'}| \times |U_3| \times |U_4|$$

$$= 11 \times 11 \times 1 \times 1$$

Thus,  $|\Delta_5| = 121$ .

$$\Delta_6 = Orb_{G_U}\{U_1', U_2, U_3', U_4\}: U_1' \in U_1^{[2]'}, U_3' \in U_3^{[2]'}$$

$$|\Delta_6| = |U_1^{[2]'}| \times |U_2| \times |U_3^{[2]'}| \times |U_4|$$

$$= 11 \times 1 \times 11 \times 1$$

Thus,  $|\Delta_6| = 121$ .

$$\Delta_7 = Orb_{G_U}\{U_1', U_2, U_4, U_4'\}: U_1' \in U_1^{[2]'}, U_4' \in U_4^{[2]'}$$

$$|\Delta_7| = |U_1^{[2]'}| \times |U_2| \times |U_3| \times |U_4^{[2]'}|$$

$$= 11 \times 1 \times 1 \times 11$$

Thus,  $|\Delta_7| = 121$ .

$$\Delta_8 = Orb_{G_U}\{U_1, U_2', U_3', U_4\}: U_2' \in U_2^{[2]'}, U_3' \in U_3^{[2]'}$$

$$|\Delta_8| = |U_1| \times |U_2^{[2]'}| \times |U_3^{[2]'}| \times |U_4|$$

$$= 1 \times 11 \times 11 \times 1$$

Thus,  $|\Delta_8| = 121$ .

$$\Delta_9 = Orb_{G_U}\{U_1, U_2', U_3, U_4'\}: U_2' \in U_2^{[2]'}, U_4' \in U_4^{[2]'}$$

$$|\Delta_9| = |U_1| \times |U_2^{[2]'}| \times |U_3| \times |U_4^{[2]'}|$$

$$= 1 \times 11 \times 1 \times 11$$

Thus,  $|\Delta_9| = 121$ .

$$\Delta_{10} = Orb_{G_U}\{U_1, U_2, U_3', U_4'\}: U_3' \in U_3^{[2]'}, U_4' \in U_4^{[2]'}$$

$$|\Delta_{10}| = |U_1| \times |U_2| \times |U_3^{[2]'}| \times |U_4^{[2]'}|$$

$$= 1 \times 1 \times 11 \times 11$$

Thus,  $|\Delta_{10}| = 121$ .

**(d) Suborbits with exactly exactly 1 ordered  $\gamma$  –tuple of element from  $P$ .**

$$\Delta_{11} = Orb_{G_U}\{U_1, U_2', U_3', U_4'\}: U_2' \in U_2^{[2]'}, U_3' \in U_3^{[2]'}, U_4' \in U_4^{[2]'}$$

$$|\Delta_{11}| = |U_1| \times |U_2^{[2]'}| \times |U_3^{[2]'}| \times |U_4^{[2]'}|$$

$$= 1 \times 11 \times 11 \times 11$$

Thus,  $|\Delta_{11}| = 1\ 331$ .

$$\Delta_{12} = Orb_{GU}\{U_1', U_2, U_3', U_4'\} : \in U_1' \in U_1^{[2]'}, U_3' \in U_3^{[2]}, U_4' \in U_4^{[2]'}$$

$$|\Delta_{12}| = |U_1^{[2]'}| \times |U_2| \times |U_3^{[2]}| \times |U_4^{[2]'}|$$

$$= 11 \times 1 \times 11 \times 11$$

Thus,  $|\Delta_{12}| = 1\ 331$ .

$$\Delta_{13} = Orb_{GU}\{U_1', U_2', U_3, U_4'\} : \in U_1' \in U_1^{[2]'}, U_2' \in U_2^{[2]}, U_4' \in U_4^{[2]'}$$

$$|\Delta_{13}| = |U_1^{[2]'}| \times |U_2^{[2]}| \times |U_3| \times |U_4^{[2]'}|$$

$$= 11 \times 11 \times 1 \times 11$$

Thus,  $|\Delta_{13}| = 1\ 331$ .

$$\Delta_{14} = Orb_{GU}\{U_1', U_2', U_3', U_4\} : \in U_1' \in U_1^{[2]'}, U_2' \in U_2^{[2]}, U_3' \in U_3^{[2]'}$$

$$|\Delta_{14}| = |U_1^{[2]'}| \times |U_2^{[2]}| \times |U_3^{[2]'}| \times |U_4|$$

$$= 11 \times 11 \times 11 \times 1$$

Thus,  $|\Delta_{14}| = 1\ 331$ .

(e) **Suborbits with no ordered  $\gamma$  –tuple of elements from  $U$ .**

$$\Delta_{15} = Orb_{GU}\{U_1', U_2', U_3', U_4'\} : \in U_1' \in U_1^{[2]'}, U_2' \in U_2^{[2]}, U_3' \in U_3^{[2]}, U_4' \in U_4^{[2]'}$$

$$|\Delta_{15}| = |U_1^{[2]'}| \times |U_2^{[2]}| \times |U_3^{[2]}| \times |U_4^{[2]'}|$$

$$= 11 \times 11 \times 11 \times 11$$

Thus,  $|\Delta_{15}| = 14\ 641$ .

**Table 3. Subdegrees of  $A_4 \times A_4 \times A_4 \times A_4$  Acting on  $U_1^{[2]} \times U_2^{[2]} \times U_3^{[2]} \times U_4^{[2]}$**

<b>Length of suborbit</b>	<b>1</b>	<b>11</b>	<b>121</b>	<b>1 331</b>	<b>14 641</b>
<b>Number of suborbits</b>	<b>1</b>	<b>4</b>	<b>6</b>	<b>4</b>	<b>1</b>

From Theorem 2.3, we have;

$$|U_1^{[2]} \times U_2^{[2]} \times U_3^{[2]} \times U_4^{[2]}| = \left(\frac{4!}{(4-2)!}\right)^4 = 20\ 736.$$

The sum of the elements in all the 16 orbits is;

$$1 + 4(11) + 6(121) + 4(1\ 331) + 14\ 641 = 20\ 736 = |U_1^{[2]} \times U_2^{[2]} \times U_3^{[2]} \times U_4^{[2]}|.$$

### 3. Conclusion

From this research, it can be concluded that using combinatorial formula and mathematical induction, the rank of  $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$  acting on  $U_1^{[\gamma]} \times U_2^{[\gamma]} \times U_3^{[\gamma]} \times U_4^{[\gamma]}$ ,  $\forall n - \gamma \geq 2$ , is 16 and the subdegrees

are;  $1, \left(\frac{n!}{(n-\gamma)!} - 1\right), \left(\frac{n!}{(n-\gamma)!} - 1\right), \left(\frac{n!}{(n-\gamma)!} - 1\right), \left(\frac{n!}{(n-\gamma)!} - 1\right), \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^2, \left(\frac{n!}{(n-\gamma)!} - 1\right)^3, \left(\frac{n!}{(n-\gamma)!} - 1\right)^3, \left(\frac{n!}{(n-\gamma)!} - 1\right)^3, \left(\frac{n!}{(n-\gamma)!} - 1\right)^3$  and  $\left(\frac{n!}{(n-\gamma)!} - 1\right)^4$ .

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Author(s) hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

### Competing Interests

Authors have declared that no competing interests exist.

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