MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

SCHOOL OF PURE, APPLIED AND HEALTH SCIENCES

MASTER OF SCIENCE EXAMINATION

COURSE CODE: MAT 8111

COURSE TITLE: GROUP THEORY

DATE: **TIME: 3 Hours**

INSTRUCTIONS TO CANDIDATES

Answer Question **ONE** and any other **TWO** questions

*This paper consists of **THREE** printed pages. Please turn over.*

QUESTION ONE – 30 MARKS

a) Define a composition series and hence show that the group of integers $\mathbb{Z}$ has no composition series. **(3 Marks)**

b) Find a finite abelian group which is isomorphic to each of the following direct product of cyclic groups of prime power order:

i) $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

ii) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ **(2 Marks)**

c) State and prove the Jordan-Holder theorem. **(4 Marks)**

d) Given that $K \lhd G$, prove that G acts on K as a group by $k^g = g^{-1}kg \;\; \forall k \in K$ and $g \in G$

(5 Marks)

e) i) Distinguish between a lower central series and an upper central series. **(2 Marks)**

ii) Show that every nilpotent group is soluble and hence by a counter example, demonstrate that the converse is not true. **(6 Marks)**

f) Given that H and K are groups, show that $\forall h, h' \in H$ and $\forall k, k' \in K$,

i) $H \times K$ is a group **(5 Marks)**

ii) $H \times K \cong K \times H$ **(3 Marks)**

QUESTION TW0 – 15 MARKS

a) State the fundamental theorem of finitely generated abelian groups. **(2 Marks)**

b) Given that H and K are groups and H acts on K, show that the set of all ordered (h, k) with $h \in H,\; k \in K$ acquires the structure of a group by $(h_1, k_1)(h_2, k_2) = (h_1h_2,\; k_1k_2)$ $\forall h_1, h_2 \in H$ and $\forall k_1, k_2 \in K$. **(6 Marks)**

c) If m is a square free integer, then prove that every abelian group of order m is cyclic.

(3 Marks)

d) Find all abelian groups of order 360 up to isomorphism. **(4 Marks)**

QUESTION THREE – 15 MARKS

a) i) Differentiate between external direct product and internal direct product. **(2 Marks)**

ii) Prove that the external direct product of H and K is abelian if and only if H and K are abelian groups. **(5 Marks)**

b) Find the order of $(4,8)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{10}$. **(3 Marks)**

c) Let n and m be relatively prime. Show that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$. **(5 Marks)**

QUESTION FOUR – 15 MARKS

a) i) Define a soluble group. **(1 Mark)**

ii) Prove that a finite group G is soluble if it contains a normal subgroup K such that K and G/K are soluble. **(5 Marks)**

b) Construct an ascending series of a group G. **(4 Marks)**

c) Define a p - group and hence show that every p -group is nilpotent. **(5 Marks)**

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