UNIVERSITY EXAMINATIONS, 2024
FOURTH YEAR EXAMINATION FOR
THE DEGREE OF BACHELOR OF MATHEMATICS
MAT:4235-1: PARTIAL DIFFERENTIAL EQUATIONS

IIInstructions to candidates:

Answer Question 1. And any other TWO.

All Symbols have their usual meaning

DATE: 2024 TIME:2hrs

Question 1(Entire course: 20 Marks)

(a) **(Diffusion Equation)** Consider the scalar KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{on} \quad \mathbb{R} \times (0, \infty). \tag{1}$$

We look for a traveling wave solution of the form

$$u(x,t) = v(x - \sigma t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{2}$$

where σ is called the *speed* and v the *profile*.

(i) Show that (1) satisfies the system of first oder ODE

$$\begin{aligned} v' &= w \\ w' &= \sigma v - 3v^2, \end{aligned} \tag{3}$$

where the prime indicates the derivative with respect to $s := x - \sigma t$. (**3 Marks**)

(ii) What is the value of

$$\lim_{s \to \infty} (v, w), \quad \text{and} \quad \lim_{s \to -\infty} (v, w)?$$

(**2 Marks**)

(iii) Sketch the phase portrait for the ODE system derived above.(**3 Marks**)

(iv) Find the explicit values for v and σ. (**6 Marks**)

(b) Check the validity of the maximum principle for the harmonic function

$$u(x) := \frac{(1 - x_1^2 - x_2^2)}{(1 - 2x_1 + x_1^2 + x_2^2)}$$

in the disk $\bar{U} = \{x_1^2 + x_2^2 \leq 1\}$. Explain. (**6 Marks**)

Question 2(15 Marks) (Laplace's Equation and Harmonic Functions)

(a) **Strong Maximum Principle**.
Suppose $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ is harmonic within U. Using the mean value property for Laplace's equation, prove that:

(i)

$$\max_{\bar{U}} u = \max_{\partial U} u \tag{4}$$

(**5 Marks**)

(ii) If U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u, \tag{5}$$

then u is constant within U. Give a simple example for $U \subset \mathbb{R}$. (**5 Marks**)

(b) Let U be a bounded open set. Let $g \in \mathcal{C}(\partial U)$ and $f \in \mathcal{C}(U)$.

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U., \end{aligned} \tag{6}$$

Using energy methods, prove that (6) has at most one solution $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$. (**5 Marks**)

Question 3 (15 Marks) (Green's function)

Suppose $u \in \mathcal{C}^2(U)$ solves the Boundary Value Problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U, \end{aligned} \tag{7}$$

where f, g are given. The Representation formula using Green's function is given by

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G(x,y)}{\partial \nu} ds(y) + \int_U f(y)G(x,y)dy, \quad x \in U, \tag{8}$$

where $G(x,y), x, y \in U$ is the Green's function.

(a) Determine Green's function for the unit ball

$$U = \{x = (x_1, x_2) \in \mathbb{R}^2 | |x| < 1\}$$

(**5 Marks**)

(b) Determine

$$\frac{\partial G(x,y)}{\partial \nu}$$

for the U. (**3 Marks**)

(c) Suppose in Equation (7) $f = 0$, find the representation formula for $u(x)$ in U. (**2 Marks**)

(d) Suppose in Equation (7) $f = 0$, and $u = g(\theta)$ for $|x| = a$, find its explicit (poissons integral) solution in

$$U = \{x = (x_1, x_2) \in \mathbb{R}^2 | |x| < a\}.$$

(**5 Marks**)

Question 4 : Heat Equation (15 Marks) Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show

$$u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$$

also solves the heat equation for each $\lambda \in \mathbb{R}$ (**4 Marks**)

(b) Use (a) to show that

$$\frac{\partial u_\lambda}{\partial \lambda}|_{\lambda=1} = v(x,t) := x.Du(x,t) + 2tu_t(x,t)$$

also solves the heat equation as well. (**4 Marks**)

(c) Assume $n = 1$ and $u(x,t) = v(\frac{x^2}{t})$.

(i) Show

$$u_t = u_{xx}$$

if and only if

$$4zv''(z) + (2+z)v'(z) = 0, (z > 0), \tag{9}$$

where the prime indicates differentiation with respect to z and $z := \frac{x^2}{t}$. (**3 Marks**)

(ii) Show that the general solution of (9) is

$$v(z) = c \int_0^z e^{\frac{-s^2}{4}} s^{-1/2} ds + d, \tag{10}$$

where c and d are constants. (**2 Marks**)

(iii) Differentiate $v(\frac{x^2}{t})$ with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for n = 1. (**2 Marks**)