MAASAI MARA UNIVERSITY
REGULAR UNIVERSITY EXAMINATIONS
2023/2024 ACADEMIC YEAR
THIRD YEAR SECOND SEMESTER
SCHOOL OF PURE APPLIED AND HEALTH SCIENCES
THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

MAT 3227-1:RINGS AND MODULES

Instructions to candidates:

Answer Question 1. And any other TWO.
All Symbols have their usual meaning

DATE: TIME:

Question 1(20 Marks)

(a) Define a ring (**3 Marks**)

(b) Give an example of a non-commutative ring (**1 Mark**)

(c) Let R be the ring of 2×2 matrices over $\mathbb{Z}$. Let $A = \begin{bmatrix} 2 & 17 \\ 0 & 0 \end{bmatrix}$ Determine a non-zero element B in R such that $AB = 0$, the zero matrix. Show that there are infinitely many elements X in R such that $AX = 0$ (**3 Marks**)

(d) i Give the definition of a two sided ideal of a ring R (**2 Marks**)

ii Prove that $3\mathbb{Z}$ (all multiples of 3) is an ideal of $\mathbb{Z}$ (**2 Marks**)

iii Exhibit four more ideals of $\mathbb{Z}$ (**2 Marks**)

(e) Let R be a ring. Explain the meaning of the statement M is a left R-module (**2 Marks**)

(f) Give an example of a $\mathbb{Z}$-module (**1 Mark**)

(g) Give the definition of a Euclidean ring (**2 Marks**)

(h) Give two examples of a Euclidean ring (**2 Marks**)

Question 2 (15 Marks)

(a) Prove that every ideal of a Euclidean ring is generated by one element (**5 Marks**)

(b) Determine the generator(s) of the following

i the ideal $15\mathbb{Z} + 27\mathbb{Z}$ of the ring $\mathbb{Z}$ (**2 Marks**)

ii the ideal $15\mathbb{Z} \cap 27\mathbb{Z}$ of the ring $\mathbb{Z}$ (**2 Marks**)

iii the ideal $3\mathbb{Z} + 4\mathbb{Z}$ (**2 Marks**)

(c) Draw a lattice diagram showing the relationship between the ideals $6\mathbb{Z}$, $17\mathbb{Z}$,$18\mathbb{Z}$, $4\mathbb{Z}$, $8\mathbb{Z}$ (and $\mathbb{Z}$) (**6 Marks**)

Question 3 (15 Marks)

Let M be the Abelian additive group of 3×2 matrices over $\mathbb{Z}$

(a) Write down three distinct elements of M **(1 Mark)**

(b) Show that M is an R-module where R is the ring of 2×2 matrices over $\mathbb{Z}$ **(3 Marks)**

(c) State whether M is a left or a right R-module **(1 Mark)**

(d) List 5 different non-zero R-submodules of M **(5 Marks)**

(e) Show that there are infinitely many R-submodules of M **(3 Marks)**

(f) Determine three submodules $M_1,\ M_2,\ M_3$ such that **(2 Marks)**

$$0 \neq M_1 \underset{\neq}{\subset} M_2 \underset{\neq}{\subset} M_3 \underset{\neq}{\subset} M$$

Question 4 (15 Marks)

(a) Give two examples of a commutative integral domain with identity (which is not a field). **(2 Marks)**

(b) Give an example of a maximal ideal in each of the integral domains in (a) **(2 Marks)**

(c) Prove that if I is a maximal ideal in a commutative integral domain with identity R then $\frac{R}{I}$ is a field **(5 Marks)**

(d) Exhibit a case where

- (i) $\frac{R}{I}$ is an infinite countable filed **(2 Marks)**
- (ii) $\frac{R}{I}$ is an infinite uncountable field **(2 Marks)**
- (iii) $\frac{R}{I}$ has exactly 4 elements **(2 Marks)**