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A Mathematical Model of Oxygen Dependency on Temperature and Pollutant Concentration in a River

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Abstract

We formulate a model of a set of advection diffusion partial differential equations governing the concentration of pollutant and oxygen in a river. It is assumed that the concentration of dissolved oxygen is strongly influenced by temperature gradient and the concentration of pollutant is primarily influenced by factors other than temperature, such as the rate of pollutant input into the river. We use asymptotic behavior of the solutions to show that when a river is highly polluted, a slight change in temperature leads to a hypoxia.

Mathematics Subject Classifications: 35K57, 91B76, 76-10

Keywords: Dissolved oxygen, Asymptotic solution, Hypoxia

1 Introduction

Dissolved oxygen, (DO), levels is affected by water temperature, ionic strength, dissolved solids, atmospheric pressure and other parameters, see for instance [3]. Oxygen solubility decreases as these parameters increase, reducing the amount of dissolved oxygen in water. Variations in dissolved oxygen occurs depending on the fluctuations in temperature, see for instance [2]. Dissolved oxygen concentration below 2 mg/L may adversely affect the survival of aquatic organisms. This condition is referred to as hypoxia and the waters are referred

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to as hypoxic waters, see for instance [2]. A model incorporating temperature variation takes the following form:

$$\begin{aligned}
\frac{\partial(AP)}{\partial t} &= D_p\frac{\partial^2(AP)}{\partial x^2} - \frac{\partial(AvP)}{\partial x} - K_1\frac{X}{X+k}AP + AqH(x), \qquad (1)\\
\frac{\partial(AX)}{\partial t} &= D_x\frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(AvX)}{\partial x} - K_2\frac{X}{X+k}AP + A\beta(S-X)e^{-\lambda\theta},\\
& 0 < x < L, t > 0.
\end{aligned}$$

where,

$$H(x) = \begin{cases} 1, & \text{if } 0 < x \le L,\\ 0, & \text{if otherwise,}\end{cases}$$

and is used to capture the fact that pollutant is only discharged for $x > 0$, L is the total length of the polluted part of the river (m), D_p is the dispersion coefficient of pollutant downstream $(m^2\ day^{-1})$, D_x is the dispersion coefficient of dissolved oxygen $(m^2\ day^{-1})$, v is the velocity of the water $(m\ day^{-1})$, A is the cross-sectional area of the river (m^2), K_1 is the degradation rate coefficient for pollutant (day^{-1}), K_2 is the de-aeration rate coefficient for dissolved oxygen (day^{-1}), k is the half-saturated oxygen demand concentration for pollutant decay $(kg\ m^{-3})$, β is the mass transfer of oxygen from the air to water $(m^2\ day^{-1})$, θ denotes temperature, S is the saturated oxygen concentration $(kg\ m^{-3})$, λ is a positive constant and it determines the strength of the effect of temperature, t denotes time $(days)$, $X := X(x,t)$ is the dissolved oxygen concentration, $P := P(x,t)$ is the pollutant concentration and q is the rate at which pollutant is added into the river and x is a position.

The first equation includes both rate of pollutant addition along the river and its removal by aeration. The second equation is a mass balance for oxygen. The rate of increase in the concentration of oxygen by movement from the surrounding air into the river depend on temperature and is proportional to the saturated concentration S less the concentration X. The rate at which oxygen is transfered into water from the air through the water surface, per unit area and time is given by β. Thus, the mass of oxygen that is transfered through the water surface per unit area and per unit time from the air is given by $\beta(S-X)e^{-\lambda\theta}$, where $\lambda > 0$, determines the strength of the effect of temperature θ.

Pimpunchat *et al.* [5] have analysed the steady-state of Equation (1) but without the inclusion of temperature and showed that the steady-state solution depends on parameters k and q. They also showed that the dissolved oxygen concentration requirement for survival of aquatic animals such as fish is 30% of the saturated values S.

This paper is organized as follows. In section 2, we analyse the model without dispersion and a lemma is given and a statement of the main results is presented. In section 3, we analyse the model with dispersion. In section 4, we give simulated results to confirm the analytical findings. We conclude with a discussion in section 5.

For easy of analysis of Equation (1), we do non-dimensionalization, to reduce the number of parameters and group them in a meaningful way. For this purpose, we define

$$\bar{t} := \frac{v}{L}t, \bar{x} := \frac{x}{L}, \bar{X} := X, \bar{P} := \frac{P}{S}, \bar{k} := \frac{k}{S}, \epsilon_p = \frac{D_p}{L^2}, \epsilon_x = \frac{D_x}{L^2}, \gamma = \frac{q}{S}, \alpha = \beta. \quad (2)$$

and assume that the length per unit time is equal to one; that is, $\frac{L}{v} = 1$. We drop the bars for notational brevity and thus obtain:

$$\begin{aligned} \frac{\partial P}{\partial t} &= \epsilon_p \frac{\partial^2 P}{\partial x^2} - \frac{\partial P}{\partial x} - K_1 \frac{X}{X+k} P + \gamma, \\ \frac{\partial X}{\partial t} &= \epsilon_x \frac{\partial^2 X}{\partial x^2} - \frac{\partial X}{\partial x} - K_2 \frac{X}{X+k} P + \beta(1-X)e^{-\lambda\theta}, \end{aligned} \quad (3)$$

for $0 < x < 1, t > 0$.

2 Long-term solution without dispersion

Long-term solution are contained in the steady-state which are attained when $\frac{\partial(P)}{\partial t} = \frac{\partial(X)}{\partial t} = 0$. With the assumption that the speed of the water is very high and dispersion coefficient are very small compared to speed of the water, we ignore the dispersion coefficient; that is, $D_p = 0$ and $D_x = 0$, and the system of partial differential equations in Equation (3) becomes a system of ordinary differential equations given in Equations (4).

$$\begin{aligned} \frac{dP}{dx} &= -K_1 \frac{X}{X+k} P + \gamma, \\ \frac{dX}{dx} &= -K_2 \frac{X}{X+k} P + \beta(1-X)e^{-\lambda\theta}. \end{aligned} \quad (4)$$

To find the asymptotic solutions of Equations (4), we state the following elementary and useful lemma. Its proof can be found in [1, 4].

Lemma 1. *Let* $x \in (0, \infty)$ *and* $f : [x, \infty) \to \mathbb{R}$ *be a differentiable function. If the* $\lim_{x\to\infty} f(x)$ *exists and the derivative of the function* $f(x)$, $f'(x)$ *is uniformly continuous on* (x, ∞), *then* $\lim_{x\to\infty} f'(x) = 0$

Thus, as $x \to \infty$, the asymptotic solutions of the Equations (4) is

$$(P^\star, X^\star) := \Big(\frac{\gamma}{K_1}\Big(\frac{X^\star + k}{X^\star}\Big), 1 - \frac{\gamma}{K_1}\frac{K_2}{\beta}e^{\lambda\theta}\Big). \tag{5}$$

Through an elementary linearization, we see that if $\beta > \frac{\gamma K_2 e^{\lambda\theta}}{K_1}$, then the point $(P^\star, X^\star)$ is asymptotically stable and the river maintains a sustainable DO concentration which provides a more favorable habitat.

We now look at the relationship between critical temperature and pollutant concentration.
There exist critical temperature, θ_c, beyond which oxygen concentration approaches zero; that is, $X(x) = 0$ as $x \to \infty$. As the temperature of the water increases, its ability to hold dissolved oxygen decreases. We show that there is a temperature, θ_c,

$$\theta_c := \frac{1}{\lambda}\ln\Big(\frac{K_1\beta S}{qK_2}\Big), \tag{6}$$

for which we shall have a catastrophe. If $\theta \geq \theta_c$, then oxygen levels depletes making the river ecologically dead, rendering it incapable of supporting aquatic life. We also show that when $\theta < \theta_c$, the river has sufficient dissolved oxygen to support aquatic life. Finally we demonstrate that the higher the amount of pollutant,(q), the smaller the temperature change that leads to a catastrophe. This is contained in the proposition below.

Proposition 1. *There exists a θ_c where $X^\star(\theta_c,\ q) = 0$, θ_c is monotonically decreasing with respect to q.*

Proof. Suppose $X^\star(\theta,\ q) = 0$ and $X^\star_\theta \neq 0$, then by the implicit function theorem, there exist a unique $\theta_c := \theta(q)$ such that $X^\star\Big(\theta(q),\ q\Big) = 0$.

A simple computation from $X^\star(\theta,\ q)$ defined by

$$X^\star(\theta,\ q) = 1 - \frac{qK_2}{\beta S K_1}e^{\lambda\theta}, \tag{7}$$

in Equation (5) show that

$$\theta_c = \frac{1}{\lambda}\ln\Big(\frac{SK_1\beta}{qK_2}\Big). \tag{8}$$

Clearly

$$\frac{\partial\theta_c}{\partial q} = -\frac{1}{\lambda q} < 0. \tag{9}$$

□

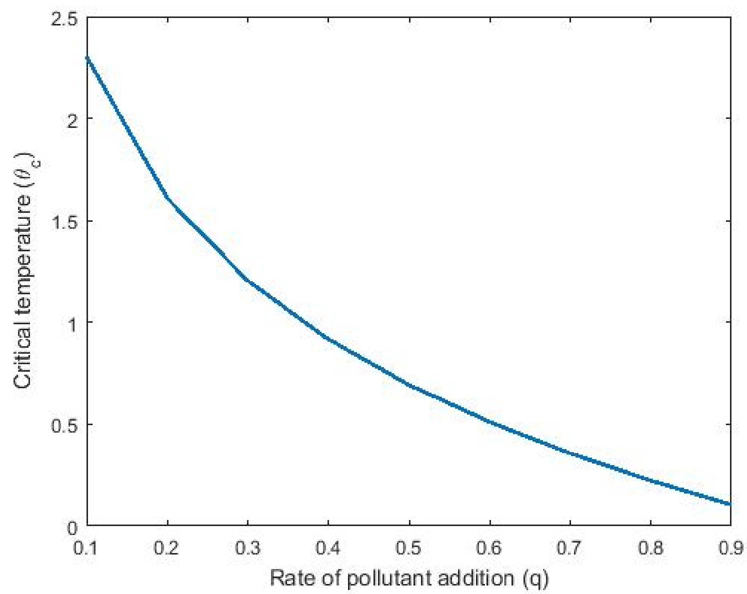

Figure 1: Critical temperature (θ_c) vs Rate of pollutant addition (q)

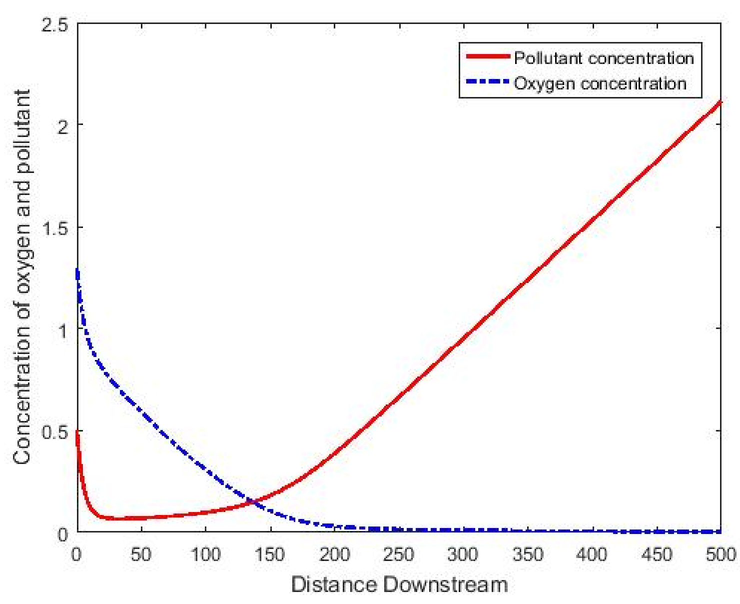

Figure 2: Oxygen and Pollutant concentration when θ is large

Graphically, the relationship between critical temperature, (θ_c), and rate of pollutant addition, (q), is as illustrated in Figure 1.

When the rate of pollutant addition, q is high, as illustrated in Figure 1, small θ_c is required for the river to be ecologically dead.

From the Figure 2, we observe that when θ is large, the concentration of oxygen depletes thus rendering the river incapable of supporting aquatic life.

3 Analytic steady-state solution for the model including dispersion ($D_p \neq 0$ and $D_x \neq 0$)

At steady state when the dispersion coefficient are included, that is $D_p \neq 0$ and $D_x \neq 0$, the second order derivative holds. The system of partial differential equation in Equation (3) becomes a system of ordinary differential equations, since they involve only one independent variable x hence Equations (10) and(11) are obtained.

$$\epsilon_p \frac{d^2P}{dx^2} - \frac{dP}{dx} - K_1 \frac{X}{X+k} P + \gamma = 0; \quad x > 0, t > 0 \tag{10}$$

$$\epsilon_x \frac{d^2X}{dx^2} - \frac{dX}{dx} - K_2 \frac{X}{X+k} P + \beta(1-X)e^{-\lambda\theta} = 0 \tag{11}$$

Since Equations (10) and (11) are a second order ODE, we can transform it to first order ODE via $P_1 := P$, $X_1 := X$, $\dot{X}_1 := X_2, \dot{X}_1 := X_2, \dot{P}_1 := P_2$, to obtain:

$$\begin{aligned}
\dot{P}_1 &= P_2, \\
\dot{P}_2 &= \alpha_1 P_2 + \alpha_2 \frac{X_1}{X_1+k} P_1 - \varphi, \\
\dot{X}_1 &= X_2, \\
\dot{X}_2 &= \alpha_3 X_2 + \alpha_4 \frac{X_1}{X_1+k} P_1 - \delta(1-X_1).
\end{aligned} \tag{12}$$

where, $\dot{P} = \frac{dP}{dx}$, $\dot{X} = \frac{dX}{dx}$, $\alpha_1 = \frac{1}{\epsilon_p}$, $\alpha_2 = \frac{K_1}{\epsilon_p}$, $\alpha_3 = \frac{1}{\epsilon_x}$, $\alpha_4 = \frac{K_2}{\epsilon_x}$, $\varphi = \frac{\gamma}{\epsilon_p}$, $\delta = \frac{\beta}{\epsilon_x} e^{-\lambda\theta}$.
Which upon simplification yields;

$$(P_1^\star, P_2^\star, X_1^\star, X_2^\star) = \Big(\frac{\gamma}{K_1}\Big(\frac{k}{X^\star} + 1\Big), 0, 1 - \frac{\gamma}{K_1}\frac{K_2}{\beta} e^{\lambda\theta}, 0\Big) \tag{13}$$

Proposition 2. *The fixed point $(P_1^\star, P_2^\star, X_1^\star, X_2^\star)$ is unstable whenever it exists.*

Proof. Evaluating the Jacobian matrix of the system (12) at $(P_1^\star, P_2^\star, X_1^\star, X_2^\star)$, yields;

$$J(P_1^\star, P_2^\star, X_1^\star, X_2^\star) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_2\phi_1 & \alpha_1 & \alpha_2\phi_2 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_4\phi_2 & 0 & \alpha_4\phi_2+\delta & \alpha_3 \end{pmatrix} \tag{14}$$

where, $\phi_1 = \left(\frac{\beta K_1 - \gamma K_2 e^{\lambda\theta}}{\beta K_1 - \gamma K_2 e^{\lambda\theta} + \beta k K_1}\right)$, $\phi_2 = \left(\frac{k\beta^2 K_1 \gamma}{(\beta K_1 - \gamma K_2 e^{\lambda\theta})(\beta K_1 - \gamma K_2 e^{\lambda\theta} + \beta k K_1)}\right)$.
The eigenvalues, μ, of $J(P_1^\star, P_2^\star, X_1^\star, X_2^\star)$ are given by

$$\mu^4 + m_1\mu^3 + m_2\mu^2 + m_3\mu + m_4 = 0, \tag{15}$$

with $m_1 = -(\alpha_1+\alpha_3)$, $m_2 = \alpha_3\alpha_1 - \alpha_4\phi_2 - \delta - \alpha_2\phi_1$, $m_3 = \alpha_1(\alpha_4\phi_2+\delta) + \alpha_2\phi_1\alpha_3$, $m_4 = \alpha_2\phi_1 - \alpha_2{\phi_2}^2\alpha_4$.

By Routh-Hurwitz criteria, we shall have stability provided; $m_1 > 0, m_3 > 0, m_4 > 0$ and $(m_1m_2 - m_3)m_3 - {m_1}^2 m_4 > 0$. It is clear that $m_1 < 0$ since $\alpha_1 > 0$, $\alpha_3 > 0$ and $m_3 > 0$, if

$$\alpha_1(\alpha_4\phi_2 + \delta) + \alpha_2\phi_1\alpha_3 > 0. \tag{16}$$

Equation (16) is satisfied when $\phi_1, \phi_2 > 0$ and is achieved when $\beta > \frac{qK_2e^{\lambda\theta}}{SK_1}$. Also $m_4 > 0$ if $\alpha_2\phi_1 > {\phi_2}^2\alpha_4\alpha_2$ and $\phi_1, \phi_2 > 0$. Since $m_1 < 0$, then the fixed point $(P_1^\star, P_2^\star, X_1^\star, X_2^\star)$ is unstable, which implies that the river ecosystem is prone to fluctuations or disturbances which poses challenges to aquatic organisms, affecting their population and the overall ecosystem dynamics due to low oxygen. □

4 Numerical simulation

We use Matlab software for numerical simulations to describe the results for Equation (3).

Figure 3 show that when the rate of pollutant addition, q is small, the concentration of pollutant in the river is low and $P(x,t) \to 0.15$ as $x \to \infty$.

Figure 4 shows that when q is high, the concentration of pollutant increases downstream as distance increases and $P(x,t) \to 4$ as $x \to \infty$.

Therefore, from Figure 3 and Figure 4, we see that pollutant concentration in a river depends on the rate of pollutant addition q.

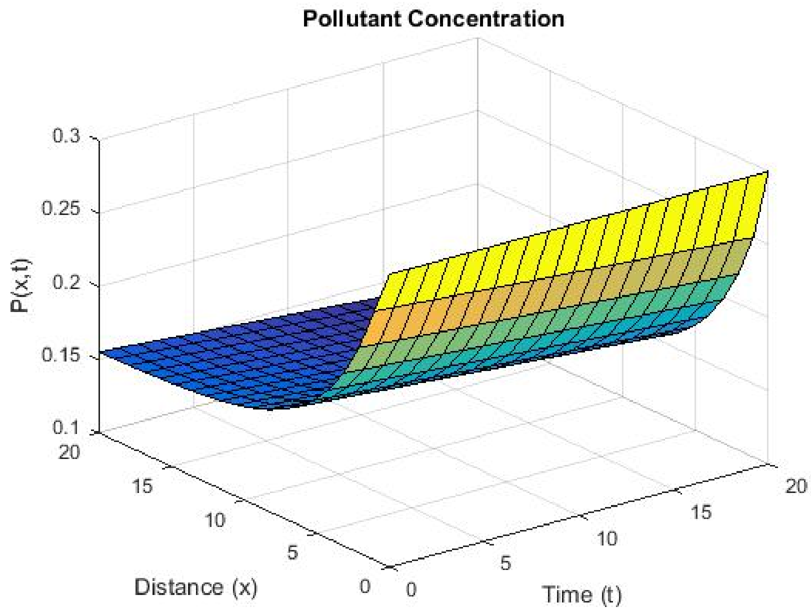

Figure 3: Pollutant concentration when q=0.05

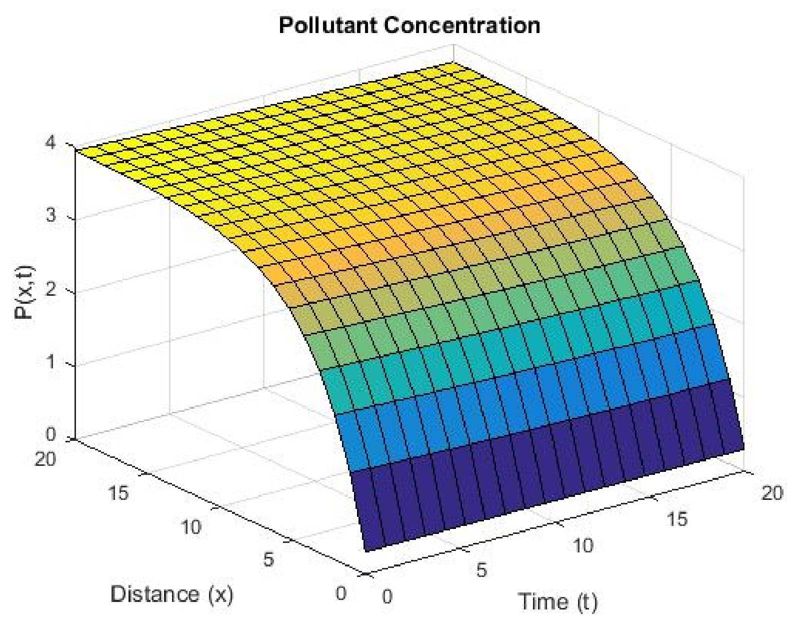

Figure 4: Pollutant Concentration when $q = 0.98$

Figure 5 shows that when the distance x increases, the concentration of oxygen slightly decreases but remains high. In this case, we see that when the temperature of the water in a river is less than the critical temperature, that

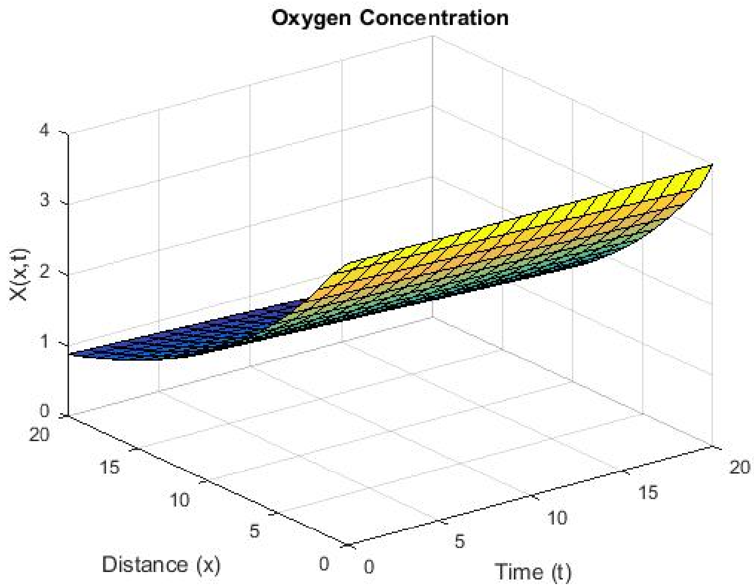

Figure 5: Oxygen Concentration when $\theta < \theta_c$

is, $\theta < \theta_c$, the concentration of oxygen is high and $X(x,t) \to 1$ as $x \to \infty$.

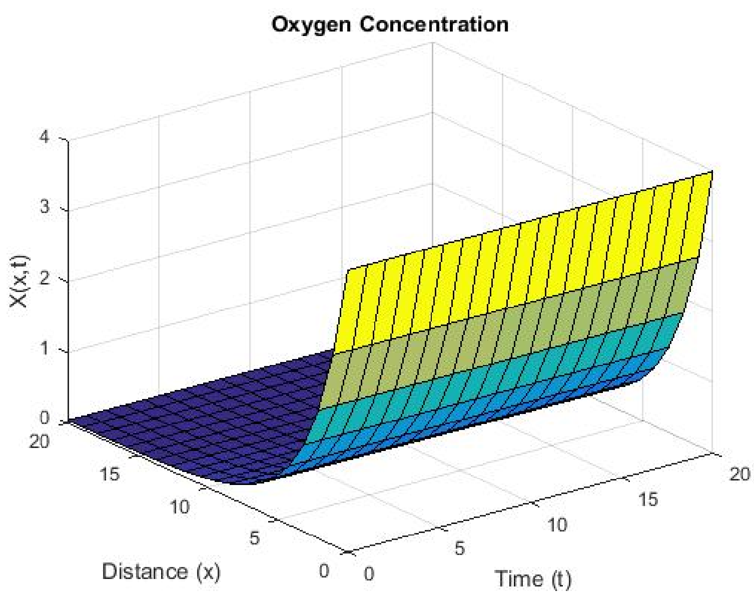

Figure 6: Oxygen concentration when $\theta_c = \theta$

Figure 6 and Figure 7 shows that when distance x increases, the concentration of oxygen approaches zero, that is $X(x,t) \to 0$ as $x \to \infty$. The decrease in the concentration of oxygen is more rapidly and significantly when $\theta = \theta_c$ and

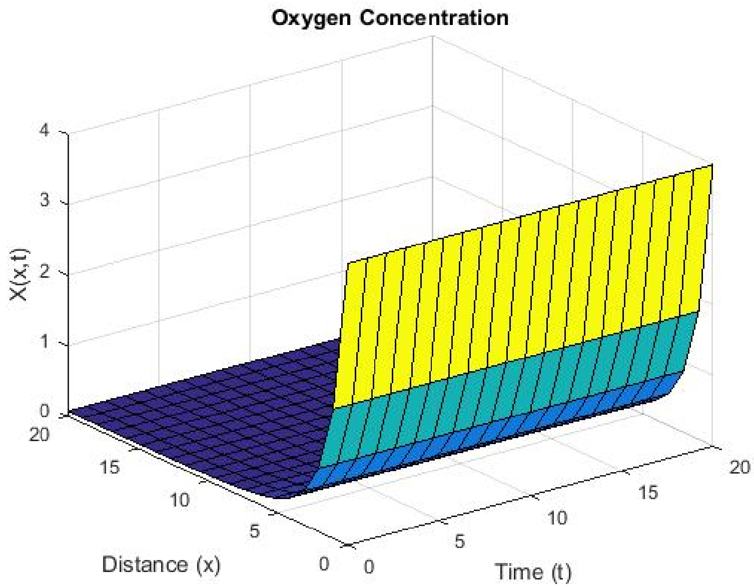

Figure 7: Oxygen concentration when $\theta > \theta_c$

when $\theta > \theta_c$. More precisely, the decrease in oxygen concentration is faster in the initial stages. Afterwards, the decline rate slows and the concentration of oxygen gradually approaches zero. Thus, in these two cases, oxygen concentration is not changing with distance as x increases as it goes to zero when $\theta = \theta_c$ and $\theta > \theta_c$ respectively.

Thus, we see from Figure 5, Figure 6 and Figure 7 that oxygen concentration in a river depends on temperature of the water. This results agrees with analytical results in the sense that, when $\theta = \theta_c$ and $\theta > \theta_c$, $X \to 0$ as distance increases.

5 Conclusion

From the model presented, we have shown that, there is a temperature, θ_c beyond which oxygen levels approach zero; that is, $X = 0$, as $x \to \infty$ and if $\theta \geq \theta_c$, then oxygen levels depletes making the river ecologically dead, rendering it incapable of supporting aquatic life. We have also shown that if $\theta < \theta_c$, then the river remains conducive to supporting aquatic life.

Furthermore, we have shown that when the river is highly polluted, a slight change in temperature leads to catastrophe rendering the river incapable of supporting aquatic life.

Therefore, it is important to monitor water temperature, oxygen levels and pollutant concentration in the river to track changes over time to ensure that oxygen concentration levels in a river remains above a critical threshold.

We recommend adaptive strategies to address extreme temperature fluctuations and their effects and reduce river pollution.

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