American Journal of Theoretical and Applied Statistics
2023; 12(4): 66-71
http://www.sciencepublishinggroup.com/j/ajtas
doi: 10.11648/j.ajtas.20231204.11
ISSN: 2326-8999 (Print); ISSN: 2326-9006 (Online)

Derivation of Two Associate PBIBD from Necessary Properties of BIBD

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To cite this article:

Troon John Benedict, Onyango Fredrick, Karanjah Anthony. Derivation of Two Associate PBIBD from Necessary Properties of BIBD. *American Journal of Theoretical and Applied Statistics*. Vol. 12, No. 4, 2023, pp. 66-71. doi: 10.11648/j.ajtas.20231204.11

Received: April 23, 2023; **Accepted**: May 13, 2023; **Published**: July 6, 2023

Abstract: One of the main shortfall of Balanced Incomplete Block Designs is that the design is not available for all parameter sets. Thus, some of the parameter sets that satisfy the necessary conditions for BIBD cannot be constructed as such. This means that the parameter sets for the designs is not possible to be constructed with a single association scheme (λ). Given that the structural difference that exist between BIBD and PBIBD is that the later do not have a single association scheme but rather m association schemes, the study believed that if the BIBD parameter sets could not be constructed as a single associations scheme then maybe if the design is broken down to more than one association scheme in form of a PBIBD then maybe such design might later be constructed using other methods as PBIBDs. The study aimed at determining whether two association scheme PBIBDs could be derived from necessary properties of BIBDs. The main reason for maiking the transformation is that for BIBD some of the designs that satisfy the necessary properties have been determined not to exist meaning that the designs could not be constructed using a single association scheme λ. Therefore, the study felt that if the designs could be broken down into 2 association scheme (λ_1 and λ_2) then such a design might be constructed as a PBIBD. The study related the necessary properties of BIBD and two association scheme PBIBD. Using the properties, the study created eight sets of linear equations and using the Gauss Jordan Elimination method, the study was able to solve for the eight unknown parameters of the PBIBD association scheme. The study was able in the end to convert a BIBD that satisfy necessary properties of BIBD into a two association scheme PBIBD that satisfy all the necessary properties of PBIBD.

Keywords: BIBD, PBIBD, Association Scheme, Necessary Properties, Derivation

1. Introduction

A Balanced Incomplete Block Design (BIBD) is an arrangement of v treatments into b blocks each of size k such that each treatment is replicated r times in the design while each pair of treatments occur together λ times in the entire design [5, 7, 8, 15].

In order for a BIBD to exist, it is always required that the parameters of the BIBD must satisfy a number of necessary conditions in order for the design to exist [1-3, 9, 10, 13, 14, 17]. First, the total number of plots in the design (bk) is always known to be equivalent to the number of treatments v times the number of replication r. Thus,

$$bk=rv \tag{1}$$

This property can always be derived by considering the total number of blocks b in the design, to be given by;

$$\begin{aligned}
b &= r\frac{\binom{v}{k}}{\binom{v-1}{k-1}} \\
&= r\frac{v!}{k!(v-k)!} \div \frac{(v-1)!}{(k-1)!(v-k)!} \\
&= r\frac{v!}{k!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!} \\
&= r\frac{v(v-1)!}{k(k-1)!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!} \\
&= \frac{rv}{k} \\
bk &= rv
\end{aligned}$$

The second property, tells us that the total number of treatments that occur together with a particular treatment is in the design $\lambda(v-1)$ must always be equal to the number of replications r times the number of plots per block less 1 plot $(k-1)$. Thus,

$$\lambda(v-1) = r(k-1) \quad (2)$$

This property also can be derived from the total number of block b in the design, that is

$$\begin{aligned} b &= \lambda \frac{\binom{v}{k}}{\binom{v-2}{k-2}} \\ &= \lambda \frac{v!}{k!(v-k)!} \div \frac{(v-2)!}{(k-2)!(v-k)!} \\ &= \lambda \frac{v!}{k!(v-k)!} \times \frac{(k-2)!(v-k)!}{(v-2)!} \\ &= \lambda \frac{v(v-1)(v-2)!}{k(k-1)(k-2)!(v-k)!} \times \frac{(k-2)!(v-k)!}{(v-2)!} \\ &= \lambda \frac{v(v-1)}{k(k-1)} \end{aligned}$$

But it is known that $b = \frac{rv}{k}$. Therefore, it can be shown that

$$\begin{aligned} \frac{rv}{k} &= \lambda \frac{v(v-1)}{k(k-1)} \\ r &= \lambda \frac{v(v-1)k}{k(k-1)v} \\ &= \lambda \frac{v-1}{k-1} \\ \lambda(v-1) &= r(k-1) \end{aligned}$$

The properties in Equations 1 and 2, are known as the fundamental or basic necessary properties of a BIBD which must always be satisfied for the design to exist [12, 22, 24]. However, the satisfaction of these properties does not guarantee existence of the design.

A Partially Balanced Incomplete Block Design (PBIBD) is an arrangements of v treatments in b blocks each of size k based on an association scheme $q \geq 2$ such that $v > k$, each treatment occurs r times in the design and if treatments v_i and v_j are i^{th} associates, then they appear together λ_i times in the design [4, 11, 19].

When a PBIBD is arranged based on an $m-association$ scheme with $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$. Then the design must satisfy the following necessary properties [6, 20, 21];

$$rv = bk$$

$$\sum_{i=1}^{m} n_i = v - 1 \quad (3)$$

$$\sum_{i=1}^{m} \lambda_i n_i = r(k-1) \quad (4)$$

$$n_k p_{ij}^k = n_i p_{jk}^i = n_j p_{ki}^j \quad (5)$$

$$\sum_{k=1}^{m} p_{jk}^i = \begin{cases} n_j - 1 & i = j \\ n_j & i \neq j \end{cases} \quad (6)$$

For a PBIBD, the set of parameters ($v, b, k, r, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m, n_1, n_2, \ldots, n_m$) are know as parameters of the first kind while the set of parameters $(P_1, P_2, \ldots, P_m)$ are known as parameters of the second kind. For a PBIBD to exist, then both parameters of first kind and second kind must exist [6, 19-21].

The study aimed at deriving two association scheme PBIBD using necessary properties of BIBD stated in Equations 1 and 2. The motive behind this study was the believe that if certain BIBDs could not be constructed as BIBDs then the single association scheme could be converted into two association scheme which could be constructed as PBIBD.

2. Derivation of Two Association Scheme PBIBD

Based on the properties of PBIBD, in order to construct a PBIBD from necessary conditions of a BIBD and still maintain some characteristics of the original BIBD, then

$$\frac{\sum_{i=1}^{m} \lambda_i n_i}{\sum_{i=1}^{m} n_i} = \lambda \quad (7)$$

This is because according to second property of a BIBD $\lambda(v-1) = r(k-1)$.

Theorem 1. *Consider a BIBD with parameters* (v, b, k, r, λ). *Such a BIBD can be transformed into a two association scheme PBIBD with parameters* $(v, b, k, r, \lambda_1, \lambda_2, n_1, n_2, p_{11}^1, p_{12}^1, p_{21}^1, p_{22}^1, p_{11}^2, p_{12}^2, p_{21}^2, p_{22}^2)$ *where*

$$\begin{aligned} n_1 &= \frac{(v-1)(\lambda_2-\lambda)}{\lambda_2-\lambda_1} \\ n_2 &= \frac{(v-1)(\lambda-\lambda_1)}{\lambda_2-\lambda_1} \\ p_{11}^1 &= \frac{n_1^2-n_1+n_2^2-n_1n_2-n_2}{n_1} \\ p_{12}^1 = p_{21}^1 &= \frac{-n_2^2+n_1n_2+n_2}{n_1} \\ p_{22}^1 &= \frac{n_2^2-n_2}{n_1} \\ p_{11}^2 &= n_1 - n_2 + 1 \\ p_{12}^2 = p_{21}^2 &= n_2 - 1 \\ p_{22}^2 &= 0 \end{aligned}$$

Proof. First we begin by solving for n_1 and n_2. According to the third property of a PBIBD

$$n_1\lambda_1 + n_2\lambda_2 = r(k-1) \quad (8)$$

But, we know that for a BIBD $\lambda(v-1) = r(k-1)$. Therefore,

$$n_1\lambda_1 + n_2\lambda_2 = \lambda(v-1) \quad (9)$$

From the second property of a PBIBD

$$n_1 + n_2 = v - 1 \quad (10)$$

$$n_1 = (v-1) - n_2 \quad (11)$$

Now substituting for n_1 in Equation 9 we get that

$$\begin{aligned} \lambda_1\big((v-1) - n_2\big) + n_2\lambda_2 &= \lambda(v-1) \\ (v-1)\lambda_1 - n_2\lambda_1 + n_2\lambda_2 &= \lambda(v-1) \\ n_2\lambda_2 - n_2\lambda_1 &= (v-1)\lambda - (v-1)\lambda_1 \\ n_2 &= \frac{(v-1)(\lambda-\lambda_1)}{\lambda_2-\lambda_1} \end{aligned}$$

Now from Equation 10 we substitute the value of n_2 so that we can get the solution for n_1

$$
\begin{aligned}
n_1 &= (v-1) - \frac{(v-1)(\lambda-\lambda_1)}{\lambda_2-\lambda_1} \\
&= (v-1)\left(1 - \frac{(\lambda-\lambda_1)}{\lambda_2-\lambda_1}\right) \\
&= (v-1)\left(\frac{(\lambda_2-\lambda_1-\lambda+\lambda_1)}{\lambda_2-\lambda_1}\right) \\
&= (v-1)\left(\frac{(\lambda_2-\lambda)}{\lambda_2-\lambda_1}\right) \\
&= \frac{(v-1)(\lambda_2-\lambda)}{\lambda_2-\lambda_1}
\end{aligned}
$$

The values of λ_1 and λ_2 are predetermined such that $\lambda_1 < \lambda < \lambda_2$ and $(v-1) \bmod (\lambda_2 - \lambda_1) \equiv 0$.

Next we determine the solutions for p_{jk}^i. In order to find the solutions for these variables we need to setup 8 systems of equations using properties 4 and 5 of the PBIBD. Using the properties, the following list of equations are generated for p_{jk}^i.

$$p_{11}^1 + p_{12}^1 = n_1 - 1 \tag{12}$$

$$p_{21}^1 + p_{22}^1 = n_2 \tag{13}$$

$$p_{11}^2 + p_{12}^2 = n_1 \tag{14}$$

$$p_{21}^2 + p_{22}^2 = n_2 - 1 \tag{15}$$

$$n_1 p_{21}^1 - n_2 p_{11}^2 = 0 \tag{16}$$

$$n_1 p_{22}^1 - n_2 p_{21}^2 = 0 \tag{17}$$

$$p_{12}^1 - p_{21}^1 = 0 \tag{18}$$

$$p_{12}^2 - p_{21}^2 = 0 \tag{19}$$

When the above system of equations is written in matrix form the following matrices are obtained

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & n_1 & 0 & 0 & -n_2 & 0 & 0 & 0 \\
0 & 0 & 0 & n_1 & 0 & 0 & -n_2 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
p_{11}^1 \\ p_{12}^1 \\ p_{21}^1 \\ p_{22}^1 \\ p_{11}^2 \\ p_{12}^2 \\ p_{21}^2 \\ p_{22}^2
\end{bmatrix}
=
\begin{bmatrix}
n_1 - 1 \\ n_2 \\ n_1 \\ n_2 - 1 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Arranging the matrices to allows solving for the unknows using Gauss Jordan Elimination method.

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & n_1 - 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & n_2 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & n_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & n_2 - 1 \\
0 & n_1 & 0 & 0 & -n_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & n_1 & 0 & 0 & -n_2 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0
\end{pmatrix}
$$

The set of linear Equations is then solved using the using Gauss Jordan Elimination method, where the augmented matrix was converted into echelon form. The resulting solution matrix was given as

$$
\begin{pmatrix}
\frac{n_1^2 - n_1 + n_2^2 - n_1 * n_2 - n_2}{n_1} - \frac{n_2}{n_1} * p_{22}^2 \\
\frac{-n_2^2 + n_1 * n_2 + n_2}{n_1} + \frac{n_2}{n_1} * p_{22}^2 \\
\frac{-n_2^2 + n_1 * n_2 + n_2}{n_1} + \frac{n_2}{n_1} * p_{22}^2 \\
\frac{n_2^2 - n_2}{n_1} - \frac{n_2}{n_1} * p_{22}^2 \\
n_1 - n_2 + 1 + p_{22}^2 \\
n_2 - 1 - p_{22}^2 \\
n_2 - 1 - p_{22}^2 \\
p_{22}^2
\end{pmatrix}
$$

Now if we let $p_{22}^2 = a$ then the solution matrix will be given as

$$
\begin{pmatrix}
\frac{n_1^2 - n_1 + n_2^2 - n_1 * n_2 - n_2}{n_1} - \frac{n_2}{n_1} * a \\
\frac{-n_2^2 + n_1 * n_2 + n_2}{n_1} + \frac{n_2}{n_1} * a \\
\frac{-n_2^2 + n_1 * n_2 + n_2}{n_1} + \frac{n_2}{n_1} * a \\
\frac{n_2^2 - n_2}{n_1} - \frac{n_2}{n_1} * a \\
n_1 - n_2 + 1 + a \\
n_2 - 1 - a \\
n_2 - 1 - a \\
a
\end{pmatrix}
\tag{20}
$$

Now based on the values of the final solution matrix in Equation 20. The values of the parameters will be given as follows.

$$
\begin{aligned}
p_{11}^1 &= \frac{n_1^2 - n_1 + n_2^2 - n_1 n_2 - n_2}{n_1} - \frac{n_2}{n_1} a \\
p_{12}^1 &= p_{21}^1 = \frac{-n_2^2 + n_1 n_2 + n_2}{n_1} + \frac{n_2}{n_1} a \\
p_{22}^1 &= \frac{n_2^2 - n_2}{n_1} + \frac{n_2}{n_1} a \\
p_{11}^2 &= n_1 - n_2 + 1 + a \\
p_{12}^2 &= p_{21}^2 = n_2 - 1 - a \\
p_{22}^2 &= a
\end{aligned}
$$

In order to find the solution for a, from the properties of PBIBD, we know that

$$
\begin{aligned}
n_2 &= p_{21}^1 + p_{22}^1 \\
&= \frac{-n_2^2 + n_1 n_2 + n_2}{n_1} + \frac{n_2}{n_1} a + \frac{n_2^2 - n_2}{n_1} + \frac{n_2}{n_1} a \\
&= \frac{-n_2^2 + n_1 n_2 + n_2 + a n_2 + n_2^2 - n_2 + a n_2}{n_1} \\
&= \frac{n_1 n_2 + 2 a n_2}{n_1} \\
n_1 n_2 &= n_1 n_2 + 2 a n_2 \\
2 a n_2 &= 0 \\
a &= 0
\end{aligned}
$$

Therefore, given that $a = 0$, then the values of the parameters for resulting PBIBD will be given as;

$$p_{11}^1 = \frac{n_1^2 - n_1 + n_2^2 - n_1 n_2 - n_2}{n_1}$$
$$p_{12}^1 = p_{21}^1 = \frac{-n_2^2 + n_1 n_2 + n_2}{n_1}$$
$$p_{22}^1 = \frac{n_2^2 - n_2}{n_1}$$
$$p_{11}^2 = n_1 - n_2 + 1$$
$$p_{12}^2 = p_{21}^2 = n_2 - 1$$
$$p_{22}^2 = 0$$

When selecting λ_1 and λ_2 we must always ensure that all the resulting parameters of second kind $p_{jk}^i < v$.

End of Proof

Example 1

Consider a symmetric BIBD with parameters $(v = b = 22, r = k = 7, \lambda = 2)$. If we select wish convert the BIBD into PBIBD with two association scheme such that $\lambda_1 = 1$ and $\lambda_2 = 4$. Then, this will result int a PBIBD with the following parameters.

$$n_1 = \frac{21(4-2)}{4-1} = 14$$

$$n_2 = \frac{21(2-1)}{4-1} = 7$$

$$p_{11}^1 = \frac{14^2 - 14 + 7^2 - (14 \times 7) - 7}{14} = 9$$

$$p_{12}^1 = p_{21}^1 = \frac{-7^2 + (14 \times 7) + 7}{14} = 4$$

$$p_{22}^1 = \frac{7^2 - 7}{14} = 3$$

$$p_{11}^2 = 14 - 7 + 1 = 8$$

$$p_{12}^2 = p_{21}^2 = 7 - 1 = 6$$

$$p_{22}^2 = 0$$

Example 2

Consider a non-symmetric BIBD with parameters $(v = 16, b = 24, r = 9, k = 6, \lambda = 3)$. If we select wish convert the BIBD into PBIBD with two association scheme such that $\lambda_1 = 2$ and $\lambda_2 = 5$. Then, this will result into a PBIBD with the following parameters.

$$n_1 = \frac{15(5-3)}{5-2} = 10$$

$$n_2 = \frac{15(3-2)}{5-2} = 5$$

$$p_{11}^1 = \frac{10^2 - 10 + 5^2 - (10 \times 5) - 5}{10} = 6$$

$$p_{12}^1 = p_{21}^1 = \frac{-5^2 + (10 \times 5) + 5}{10} = 3$$

$$p_{22}^1 = \frac{5^2 - 5}{10} = 2$$

$$p_{11}^2 = 10 - 5 + 1 = 6$$

$$p_{12}^2 = p_{21}^2 = 5 - 1 = 4$$

$$p_{22}^2 = 0$$

The results in table 1 shows a list of two association scheme PBIBDs that could be derived from BIBDs that satisfy necessary properties together with the efficiency of the resulting PBIBDs.

Table 1. *Resulting PBIBDs Generated from Necessary Properties of BIBDs.*

BIBD	PBIBD	Efficiency
v=15, b=21, r=7, k=5 $\lambda = 2$	$\lambda_1 = 1, \lambda_2 = 3, n_1 = 7, n_2 = 7, P_1 = \begin{bmatrix} 5 & 1 \\ 1 & 6 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 6 \\ 6 & 0 \end{bmatrix}$	0.8228571
v=22, b=22, r=7, k=7 $\lambda = 2$	$\lambda_1 = 1, \lambda_2 = 4, n_1 = 14, n_2 = 7, P_1 = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}, P_2 = \begin{bmatrix} 8 & 6 \\ 6 & 0 \end{bmatrix}$	0.8803521
v=21, b=28, r=8, k=6 $\lambda = 2$	$\lambda_1 = 1, \lambda_2 = 3, n_1 = 10, n_2 = 10, P_1 = \begin{bmatrix} 8 & 1 \\ 1 & 9 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 9 \\ 9 & 0 \end{bmatrix}$	0.8674569
v=29, b=29, r=8, k=8 $\lambda = 2$	$\lambda_1 = 1, \lambda_2 = 3, n_1 = 14, n_2 = 14, P_1 = \begin{bmatrix} 12 & 1 \\ 1 & 13 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 13 \\ 13 & 0 \end{bmatrix}$	0.9007241
v=16, b=24, r=9, k=6, $\lambda = 3$	$\lambda_1 = 1, \lambda_2 = 7, n_1 = 10, n_2 = 5, P_1 = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}, P_2 = \begin{bmatrix} 6 & 4 \\ 4 & 0 \end{bmatrix}$	0.8465608
v=31, b=31, r=10, k=10 $\lambda = 3$	$\lambda_1 = 2, \lambda_2 = 4, n_1 = 15, n_2 = 15, P_1 = \begin{bmatrix} 13 & 1 \\ 1 & 14 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 14 \\ 14 & 0 \end{bmatrix}$	0.927395
v=34, b=34, r=12, k=12 $\lambda = 4$	$\lambda_1 = 3, \lambda_2 = 6, n_1 = 22, n_2 = 11, P_1 = \begin{bmatrix} 15 & 6 \\ 6 & 5 \end{bmatrix}, P_2 = \begin{bmatrix} 12 & 10 \\ 10 & 0 \end{bmatrix}$	0.9412752
v=46, b=69, r=9, k=6 $\lambda = 1$	$\lambda_1 = 0, \lambda_2 = 3, n_1 = 30, n_2 = 15, P_1 = \begin{bmatrix} 21 & 8 \\ 8 & 7 \end{bmatrix}, P_2 = \begin{bmatrix} 16 & 14 \\ 14 & 0 \end{bmatrix}$	0.825641
v=43, b=43, r=15, k=15 $\lambda = 5$	$\lambda_1 = 4, \lambda_2 = 6, n_1 = 21, n_2 = 21, P_1 = \begin{bmatrix} 19 & 1 \\ 1 & 20 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 20 \\ 20 & 0 \end{bmatrix}$	0.9548002

3. Conclusion

In conclusion, the study was able to illustrate that a two association scheme PBIBD could be derived from the necessary properties of BIBD. This means that the single association scheme that exist in BIBD could be broken down into a two association scheme making it possible to construct the BIBDs as PBIBDs. However, the study was not able to construct the derived PBIBD or show that such PBIBDs could not be constructed in their derived forms.

4. Recommendation

The study recommends that further research be carried out on how the resulting PBIBDs could be constructed or not be constructed. The ability of the design to be constructed will be a great breakthrough as it will give an alternative means dealing with non-existing BIBDs. The proof that the designs does not exist will also be a great break through as it will aid in generalizing the non-existence property for PBIBDs.

Appendix

R Code for Transforming BIBD into PBIBD

```
pbibdt2 <- function(v, b, r, k,lambda){
 for(lambda1 in seq(0,lambda-1,1)){
  for(lambda2 in seq(lambda+1,r-1,1)){
   if((((v - 1) * (lambda2 - lambda)) %% (lambda2 -
lambda1)==0) &&
     (((v - 1) * (lambda - lambda1)) %% (lambda2 -
lambda1))==0){
     # Compute n1 and n2
     n1 <- ((v - 1) * (lambda2 - lambda)) / (lambda2 -
lambda1)
     n2 <- ((v - 1) * (lambda - lambda1)) / (lambda2 -
lambda1)
     if( ((n1^2) - n1 + (n2^2) - (n1 * n2) - n2) %% n1==0){
      p111 <- ((n1^2) - n1 + (n2^2) - (n1 * n2) - n2) / n1
      if((- (n2^2) + (n1 * n2) + n2) %% n1==0){
       p112 <- (- (n2^2) + (n1 * n2) + n2) / n1
       p121 <- (- (n2^2) + (n1 * n2) + n2) / n1
       if(((n2^2) - n2) %% n1==0){
        p122 <- ((n2^2) - n2) / n1
        p211 <- n1 - n2 + 1
        p212 <- n2 - 1
        p221 <- n2 - 1
        p222 <- 0

        # Create matrices P1 and P2
        P1 <- matrix(c(p111, p112, p121, p122),
         nrow = 2, ncol = 2, byrow = TRUE)
        P2 <- matrix(c(p211, p212, p221, p222),
         nrow = 2, ncol = 2, byrow = TRUE)

        if(p111>=0 && p112>=0 && p122>=0 &&
p211>=0
        && p212>=0 && p222>=0 ){
         # Print results
         cat("v = ", v, "\tb = ", b, "\tr = ",
          r, "\tk = ", k)
         cat("\nlambda1 = ", lambda1, "\tlambda2 = ",
          lambda2, "\tlambda = ", lambda)
         cat("\nn1 = ", n1, "\tn2 = ", n2)
         cat("\nP1 = \n")
         print(P1)
         cat("P2 = \n")
         print(P2)
         library(PBIBD)
         comment1=verify(v=v,b=b,r=r,k=k,n=c(n1,n2),
         l=c(lambda1,lambda2),P=list(P1,P2))
         print(comment1)
         design1=apbibd(v=v,r=r,k=k,n=c(n1,n2),
         l=c(lambda1,lambda2),P=list(P1,P2))
         cat("\n The Efficiency of the Design is \n")
         print(design1$E)
        }
       }
      }
     }
    }
   }
  }
}
```

References

[1] Akra, U. P., Akpan, S. S., Ugbe, T. A. and Ntekim, O. E. (2021). *Finite Euclidean Geometry Approach for Constructing Balanced Incomplete Block Design (BIBD)*. Asian Journal of Probability and Statistics. 11 (4): 47-59.

[2] Alabi, M. A. (2018). *Construction of balanced incomplete block design of lattice series I and II*. International Journal of Innovative Scientific and Engineering Technologies Research. 2018; 6 (4): 10-22.

[3] Alam, N. M. (2014). *On Some Methods of Construction of Block Designs*. I. A. S. R. I, Library Avenue, New Delhi-110012.

[4] Arasu, K. T., Bansal, P. and Watson, C. (2013). Partially Balanced Incomplete Block Design with Two Associate Classes. Journal of Statistical Planning and Inference 143; 983–991.

[5] Bose, R. C. (1939), *On the construction of balanced incomplete block designs*. Annals of Eugenics, Vol. 9, pp. 353–399.

[6] Bose, R. C. and Shimamoto, T. (1952). *Classification and analysis of partially balanced incomplete block designs with two associate classes*, J. Amer. Stat. Assoc, Jfl, 151-184.

[7] Bose, R. C., Shrikhande, S. S., and Parker, E. T. (1960). Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture. Canadian Journal of Mathematics, 12, 189-203.

[8] Dey, A. (2010). *Incomplete Block Designs*. World Scientific Publishing Co. Pte. Ltd. Warren Street. U.S.A.

[9] Greig, M., and Rees, D. H. (2003). Existence of balanced incomplete block designs for many sets of treatments. Discrete Mathematics, 261 (1-3), 299-324.

[10] Goud T. S. and Bhatra, C. N. Ch. (2016). Construction of Balanced Incomplete Block Designs. International Journal of Mathematics and Statistics Invention. 4 (1) 2321-4767.

[11] Hinkelmann, K. and Kempthorne, O. (2005). *Design and Analysis of Experiments*. John Wiley and Sons, Inc., Hoboken, New Jersey.

[12] Hsiao-Lih, J., Tai-Chang, H. and Babul, M. H. (2007). *A study of methods for construction of balanced incomplete block design*. Journal of Discrete Mathematical Sciences and Cryptography Vol. 10 (2007), No. 2, pp. 227–243.

[13] Janardan, M. (2018). *Construction of balanced incomplete block design: an application of galois field*. Open Science Journal of Statistics and Application. 2013; 5 (3): 32-39.

[14] Kageyama, S. (1980). *On properties of efficiency balanced designs*. Communication in Statistics, A 9 (6), 597-616.

[15] Mahanta, J. (2018). *Construction of balanced incomplete block design: An application of Galois field*. Open Science Journal of Statistics and Application.

[16] Mandal, B. N. (2015). *Linear Integer Programming Approach to Construction of Balanced Incomplete Block Designs*. Communications in Statistics - Simulation and Computation, 44: 6, 1405-1411, DOI: 10.1080/03610918.2013.821482.

[17] Montgomery, D. C. (2019). *Design and analysis of experiment*. John Wiley and Sons, New York.

[18] Neil, J. S. (2010). *Construction of balanced incomplete block design*. Journal of Statistics and Probability. 12 (5); 231-343.

[19] Sinha, K. (1987). *Generalized Partially Balanced Incomplete Block Designs*. Discrete Mathematics 67; 315-318.

[20] Shih, Y. (1969). *Partially Balanced Incomplete Block Designs*. All Graduate Plan B and other Reports. 1133. https://digitalcommons.usu.edu/gradreports/1133

[21] Manohar, N. V. (1959). The non-existence of certain PBIB designs, Ann. Math. Stat. 30, 1051-1062.

[22] Wan, Z. X. (2009). *Design theory*. World Scientific Publishing Company.

[23] Yasmin, F., Ahmed, R. and Akhtar, M. (2015). Construction of Balanced Incomplete Block Designs Using Cyclic Shifts. Communications in Statistics—Simulation and Computation 44: 525-532. DOI: 10.1080/03610918.2013.784984.

[24] Yates, F. (1936). *A new method of arranging variety trials involving a large number of varieties*. J. Agric. Sci., 26, 424-445.