UNIVERSITY EXAMINATIONS, 2019
THIRD YEAR EXAMINATION
FOR
THE DEGREE OF BACHELOR OF SCIENCE
MAT:3222- ORDINARY DIFFERENTIAL EQUATIONS II

Instructions to candidates:

Answer Question 1. And any other TWO.
All Symbols have their usual meaning

DATE: − − − − − 2019 TIME:——- to ——-.

Question 1(Entire course: 30 Marks)

(a) Let $A \in L(\mathbb{R}^n)$. Show that the unique solution of the initial value problem

$$\dot{x} = Ax, \quad x(0) = K \in \mathbb{R}^n \tag{1}$$

is $e^{tA}K$. **(4 Marks)**

(b) Find the solution $x(t, x_0)$ of the initial value problem

$$\dot{x} = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} x, \quad x(0) = (K_1, K_2, 0),$$

where K_1 and K_2 are nonzero constants . **(3 Marks)**
What is the value of

$$\lim_{t\to\infty} |x(t, x(0))|?$$

(1 Mark)

(c) Consider the equation

$$\dot{x} = (x-2)(x+1). \tag{2}$$

Sketch on the same diagram the possible solution curves for Equation(2) for the following initial conditions

- $-\infty < x(0) < -1$ **(1 Mark)**
- $x(0) = -1$ **(1 Mark)**
- $-1 < x(0) < 2$ **(1 Mark)**
- $x(0) = 2$ **(1 Mark)**
- $2 < x(0) < \infty$ **(1 Mark)**

(d) Consider the differential equation $\dot{x} = Ax$, where $A \in R^{2\times 2}$ constant matrix. By using the determinant and trace of A, determine under what conditions will one expect to get a stable or unstable

(i) Nodes, **(2 Marks)**

(ii) Saddles, **(2 Marks)**

(iii) Centers, **(1 Mark)**

(iv) Foci at the origin. **(2 Marks)**

(e) Consider the initial value problem

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{3}$$

where $f \in \mathcal{C}(D, \mathbb{R}) \quad D \subset I \times \mathbf{R}$. Prove that $x(t)$ is a solution of this initial value problem for $t \in I$ **if and only if** $x(t)$ is a continuous function that satisfies the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))ds, \qquad \text{for all } t \in I \tag{4}$$

(**10 Marks**)

Question 2 (Linear System: 20 Marks) Consider the nonhomogeneous initial value problem

$$x\dot(t) = Ax(t) + b(t), \quad x(0) = x_0, \tag{5}$$

where $A \in \mathbf{R}^{n\times n}, \quad b(t) \in \mathbf{R}^n$ are continuous functions, and other symbols have their usual meaning.

(a) What do you understand by $X(t) \in \mathbf{R}^{n\times n}$ is a fundamental solution of the differential equation (5) with $b(t) = 0$. (**3 Marks**)

(b) Derive the *variation-of-constants formula* for Equation (5) above. (**6 Marks**)

(c) By using the method in (b) above, solve the system of equations below

$$\begin{aligned} \dot{x_1} &= x_1 + x_2 + t, \\ \dot{x_2} &= -x_2 + 1, \end{aligned} \tag{6}$$

subject to $x_1(0) = 1, \ x_2(0) = 0$. (**11 Marks**)

Question 2 (Two species competing for the same prey: 20 Marks) Consider the differential equations that model the populations $x_1(t)$ and $x_2(t)$ at a time $t \geq 0$ of two competing species

$$\begin{aligned} \dot{x_1} &= ax_1(1 - x_1) - bx_1x_2 \\ \dot{x_2} &= cx_2(1 - x_2) - dx_1x_2 \end{aligned} \tag{7}$$

Let $a = 1, b = 2, c = 1$, and $d = 3$.

(i) On one phase plane, sketch the iscolines of the differential equations (7) and determine **all** its equilibrium points. (**4 Marks**)

(ii) Determine the type of stability of all the equilibrium points in (i) above. (**6 Marks**)

(iii) Sketch the phase plane and clearly indicate the directions of the vector field defined by Equation(7). (**2 Marks**)

(iv) State algebraically and sketch by shading appropriately the basin of attraction of each attracting fixed point. (**4 Marks**)

(v) What are the likely populations of the species in the long-term. State the reason(s) for the choice of your answer. (**2 Marks**)

(vi) If $a = 3, b = 2, c = 4$, and $d = 3$, show that the populations coexist at some point $\bar{x}(\frac{2}{3}, \frac{1}{2})$. (**2 Marks**)

Question 4 (20 Marks): Simple HIV/AIDS Model We are interested in the development of a simple AIDS epidemic model in a heterosexual population of adults. Let a population be divided into three categories $S(t), I(t), A(t)$ as defined below:

$S(t)$: Susceptibles, the number of individuals at time t not yet infected but may, if exposed to the diseases.

$I(t)$: Infectives, the number of individuals at time t who are already infected with HIV/AIDS and are capable of transmitting the virus.

$A(t)$: The number of individuals who hve developed full-blown AIDS sypmtoms at time t.

μ: The par capita AIDS- nonrelated mortality rate.

d: The rate at which AIDS patient are dying due to AIDS causes.

v: The rate at which HIV infected (infectives) progress to AIDS.

λ: The probability of getting infected by HIV/AIDS from a randomly chosen partner.

c: The rate at which an individual acquires new (changes) sexual partners.

β: The transmission probability; that is, the probability of getting infected from a partner.

B: Recruitment rate of susceptibles into a population.

The model

We make the following assumptions.

The recruitment into the population of study (sexually mature adults) is mainly by birth.

The full-blown AIDS cases are easily recognized in the population and are no longer a threat in the spread of the epidemic; that is, they do not participate in the population dynamics.

An individual once infected becomes and remains infective until death.

The force of infection depends on the number of infectives in the population and the product βc.

We consider a homogeneous population (uniform mixing).

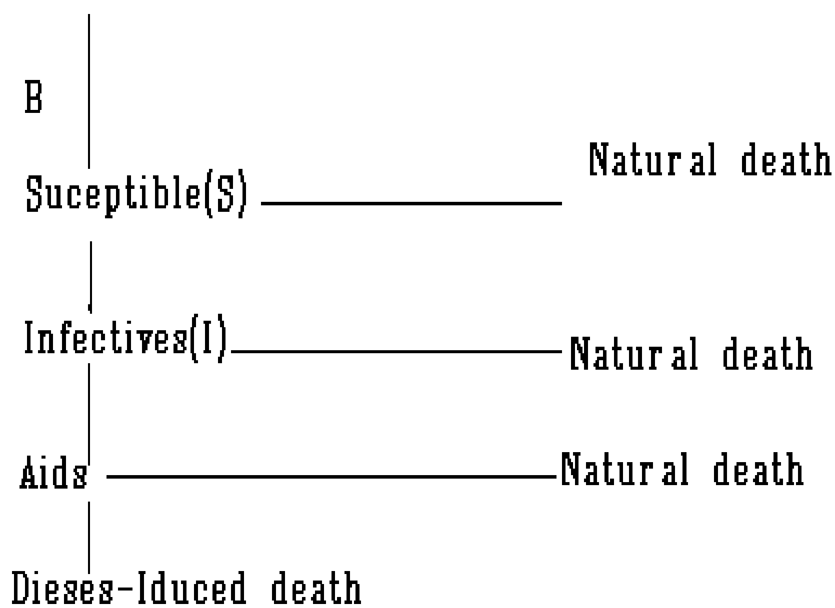

Figure 1: A Schematic Model for a siple AIDS Model

A reasonable first model, based on the flow diagram is

$$\begin{aligned} \dot{S} &= B - \mu S - \lambda c S, \quad \lambda = \frac{\beta I}{N}, \\ \dot{I} &= \lambda c S - (\mu + v) I, \\ \dot{A} &= v I - (d + \mu) A, \end{aligned} \tag{8}$$

where $\frac{1}{v}$ a constant, is the average incubation time of the disease, and $N(t) = S(t) + I(t)$.

(a)(i) What is the interpretation of $\frac{I}{S+I}$? (**2 Marks**)

(ii) If an individual of full-blown AIDS dies within 9 months to 12 months, state the interval of existence of the parameter d (**2 Marks**)

(iii) If the incubation period is 8 months, what is the value of v? (**1 Mark**)

(iv) From Equation(8), write down an expression for the basic reproductive rate of the infection R_0. (**2 Marks**)

(b) In Equation(8), if at $t = 0$, an infected individuai is introduced into an otherwise infection free population of susceptibles, we have initially $S \approx N$. Since the average incubation time from infection to development of the disease, is very much shorter than the average life expectancy of a susceptible; that is, $v \gg \mu$, we have that near $t = 0$,

$$\dot{I} \approx (\beta c - v - \mu)I \approx v(R_0 - 1)I. \tag{9}$$

(i) What is the expression for the reproductive rate of the infection R_0 described in Equation(9). (**2 Marks**)

(ii) Write an expression for the solution of Equation (9) if the initial population is $I(0)$. (**2 Marks**)

(iii) From your solution in (b) (ii), determine an expression for the doubling time of the infectives. (**2 Marks**)

(c) We notice that R_0 depends on β, c, both of which are social factors. What could one do to keep R_0 small and hence lower the rate of increase of $I(t)$?(**2 Marks**)

(d) Suppose that once an individual is tested HIV positive is exposed an thus avoided sexually, what modification could one introduce to Equation(8) to carter for this variation? (**2 Marks**)

(e) Suppose the life expectancy of children is $\frac{1}{\mu_1}$, what proportion of children born, $\tau > 0$ years ago can reach sexual maturity age τ years. (**3 Marks**)