UNIVERSITY EXAMINATIONS, 2019
FOURTH YEAR EXAMINATION FOR
THE DEGREE OF BACHELOR OF MATHEMATICS
MAT:419- PARTIAL DIFFERENTIAL EQUATIONS II

Instructions to candidates:

Answer Question 1. And any other TWO.

All Symbols have their usual meaning

DATE: 2019 TIME:2hrs

Question 1(Entire course: 30 Marks)

(a) **(Transport Equation)** Write down an explicit formula for a function u solving the initial value problem

$$\begin{aligned} u_t + b.Du + cu = 0 \quad & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g \quad & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned} \tag{1}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants. (**4 Marks**)

(b) **(Traveling Wave)** Consider the Korteweg-de vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0) \quad \text{in } \mathbb{R} \times (0,\infty). \tag{2}$$

We look for a traveling wave solution of the form

$$u(x,t) = v(x - \sigma t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{3}$$

where σ is called the *speed* and v the *profile*.

(i) Show that

$$-\sigma v + 3v^2 + v'' = 0, \tag{4}$$

where the prime indicates the derivative with respect to $\eta := x - \sigma t$. (**5 Marks**)

(ii) Write Equation(4) as a system of first oder equations involving (v, w), where $w := v'$. (**2 Marks**)

(iii) Sketch the phase phase portrait for the system in (ii) and hence determine

$$\lim_{\eta\to\infty}(v,w), \quad \text{and} \quad \lim_{\eta\to-\infty}(v,w)?$$

(**4 Marks**)

(iv) Solve explicitly Equation(4) (**5 Marks**)

(v) Sketch the wave profile. (**3 Marks**)

(c) **(Wave Equation)** Solve the initial value problem:

$$\begin{aligned} u_{tt} - \Delta u &= xt \quad , \text{ in } \mathbb{R}, t > 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= 0. \end{aligned}$$

(**4 Marks**)

(d) Let $U \subset \mathbb{R}^n$ be open and bounded, and ∂U be $\mathcal{C}^1$. Suppose that u is a harmonic function in the disk $U = \{r < 2\}$ and $r = (x_1^2 + x_2^2)^{1/2}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions
(i) Find the maximum value of u in $\bar{U}$ (**1 Marks**)
(ii) Calculate the value of u at the origin (**2 Marks**)

Question 2 (20 Marks) (Fourier Transformation) The Fourier transform on L^1.
If $u \in L^1(\mathbb{R}^n)$, we define its Fourier transform thus

$$\hat{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix.y} u(x)dx, \quad y \in \mathbb{R}^n \tag{5}$$

and its inverse Fourier transform thus

$$\check{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix.y} u(x)dx, \quad y \in \mathbb{R}^n. \tag{6}$$

These integrals converge for each $y \in \mathbb{R}^n$.

(a) **(Properties of Fourier transform).**
Assume $u, v \in L^2(\mathbb{R}^n)$, prove that

(i) $(\hat{D^\alpha u}) = (iy)^\alpha \hat{u}$ for each multiindex α such that $D^\alpha u \in L^2(\mathbb{R}^n)$. (**3 Marks**)

(ii) If $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $(u \hat{*} v) = (2\pi)^{n/2} \hat{u}\hat{v}$. (**3 Marks**)

(b) **Heat equation.**Consider the initial value problem

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned} \tag{7}$$

Find the Fourier transform $\hat{u}$ of (7) with respect to spatial variables x only.(**2 Marks**)
Solve the equation for $\hat{u}$. (**1 Mark**)
Find

$$u(x,t), \quad \text{for } x \in R^n, t > 0. \tag{8}$$

(**5 Marks**)

(c) **Telegraph equation**
The initial-value problem for the one-dimensional *telegraph equation* is

$$\begin{aligned} u_{tt} + 2du_t - u_{xx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u &= g \quad , u_t = h \text{on } \mathbb{R} \times \{t = 0\} \end{aligned} \tag{9}$$

for $d > 0$. Use Fourier transform method to obtain the representation formula:

$$\begin{aligned} u(x,t) &= \frac{e^{-dt}}{(2\pi)^{1/2}} \int_{\{y \le |d|\}} \beta_1(y) e^{ixy+\gamma(y)t} + \beta_2(y) e^{ixy-\gamma(y)t} dy \\ &+ \frac{e^{-dt}}{(2\pi)^{1/2}} \int_{\{y \ge |d|\}} \beta_1(y) e^{ixy+\delta(y)t} + \beta_2(y) e^{ixy-\delta(y)t} dy, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \gamma(y) &= (d^2 - |y|^2)^{1/2}, \quad |y| \le d \\ \delta(y) &= (d^2 - |y|^2)^{1/2}, \quad |y| \ge d \\ \hat{g}(y) &= \beta_1(y) + \beta_2(y) \end{aligned} \tag{11}$$

(**6 Marks**)

Question 3 (20 Marks) (Green's function)

Suppose $u \in \mathcal{C}^2(U)$ solves the Boundary Value Problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } U \\ u &= g \qquad \text{on } \partial U, \end{aligned} \tag{12}$$

where f, g are given. The Representation formula using Green's function is given by

$$u(x) = -\int_{\partial U} g(y) \frac{\partial G(x,y)}{\partial \nu} ds(y) - \int_U f(y) G(x,y) dy, \quad x \in U, \tag{13}$$

where $G(x,y), x, y \in U$ is the Green's function.

(a) Determine Green's function for half space

$$\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_n > 0\}$$

(**6 Marks**)

(b) Determine

$$\frac{\partial G(x,y)}{\partial \nu}$$

for the $\mathbb{R}^n_+$ (**6 Marks**)

(c) Suppose in Equation (12) $f = 0$, find the representation formula for $u(x)$ in $\mathbb{R}^n_+$ (**3 Marks**)

(d) Suppose in Equation (12) $f = 0$, $n = 2$ and $u(x_1, 0) = 1$, find its explicit solution. (**5 Marks**)

Question 4 (20 Marks) (Laplace's Equation and Harmonic Functions)

(a) **(Mean-Value formulas for Laplace's Equation).**
Theorem (Mean-value formulas for Laplace's equation). If $u \in C^2(U)$ is harmonic, then

$$u(x) = ⨍_{\partial B(x,r)} u \, dS = ⨍_{B(x,r)} u \, dy \tag{14}$$

for each ball $B(x,r) \subset U$, where U is an open set of $\mathbb{R}^n$.
Without proofing this theorem, give its physical interpretation and give a simple example for $U \in \mathbb{R}$. **(6 Marks)**

(b) **Strong Maximum Principle**.
Suppose $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ is harmonic within U. Using the Maximum principle, prove that:

(i)

$$\max_{\bar{U}} u = \max_{\partial U} u \tag{15}$$

(2 Marks)

(ii) If U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u, \tag{16}$$

then u is constant within U. **(6 Marks)**

(c)

$$\begin{aligned} -\Delta u &= f \qquad \text{in } U \quad f \in \mathcal{C}(U) \\ u &= g \qquad \text{on } \partial U, \quad g \in \mathcal{C}(\partial U) \end{aligned} \tag{17}$$

Use the maximum principal to prove that Equation(17) has at most one solution,$u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$. **(6 Marks)**